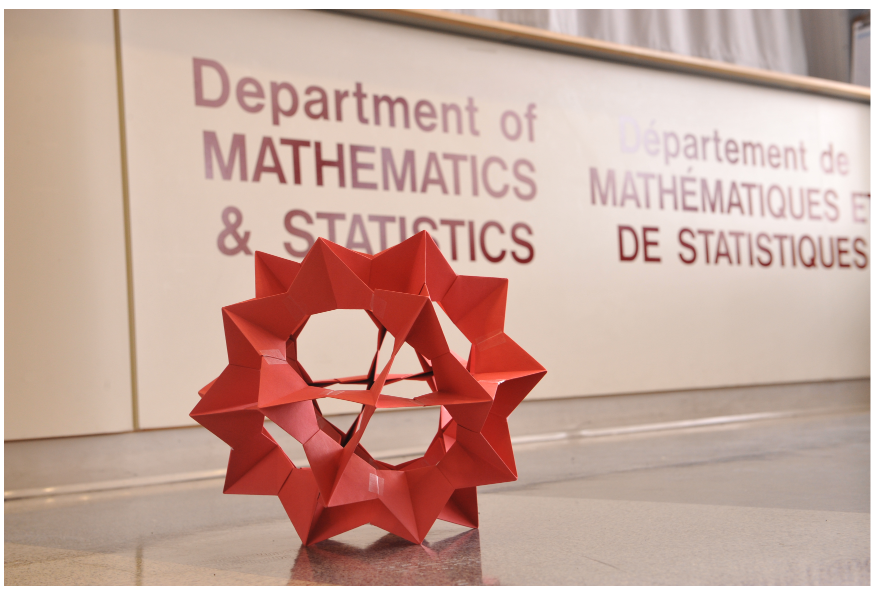
UNIVERSITÉ Concordia UNIVERSITY

MATH 200



Preface

This manual has been prepared for the introductory course **Math 200**, offered by Department of Mathematics and Statistics at Concordia University and it is intended to cover the algebraic operations and solutions to linear, quadratic and rational equations. It also has a couple of topics on linear inequalities, introduction to functions and graphing linear functions.
The book starts with a section on operations on real numbers and the order of operations. Since the students of this course are assumed to have a good knowledge of this topic, it is included as "Prealgebra Review" in Chapter 0 for those who want to refresh their memories. Therefore, the students may start studying from Chapter 1 entitled "Operations on Algebraic Expressions" as the first main topic in the course Math 200.
In this book, it has been tried to keep the order of the topics presented in the 12 lessons of this course. The registered students can find those lesson in the form of videos on Econcordia Website of the course. There is, however, one except which are the topics covered in Lesson 9. This lesson covering two topics is included in Chapter 1 under the titles "Integer Exponents" and "Roots and Radicals". The reason for this displacement is that these topics are more relevant to the material of the first chapter. Moreover, in some problems and exercises of Chapters 1, 2 and 3, students will need to work with operations on integer and fractional exponents.
In each of the Chapters 2, 7 and 8, one can find a couple of related lessons, also in the same order as they are presented in the course website.
There are also a couple of sections in this manual that you may not find in an specific lesson of the course, and yet the materials of these sections are covered or mentioned in different lessons. That is why we have decided to include those sections in the manual. Even though it is to the discretion of the instructor to decide how and to what depth they will cover the materials, examples and exercises of the manual.

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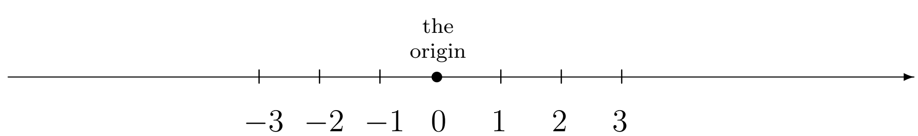
Contents

Chapter 0
Prealgebra Review

0.1 Sets of Numbers and Their Operations

Algebraic Expressions are, to a large extend, the generalization to symbols and letters of the arithmetic of the real numbers. So, let us first recall some important number sets:

- The **natural** numbers:
$$\mathbb{N} = \{1, 2, 3, ...\};$$
- The **integers**, or as some call them, the whole numbers:
$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\};$$
- The **rational** numbers:
$$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\};$$
- The **real** numbers: $\mathbb{R}$ which includes all the rational as well as **irrational** numbers; every real number corresponds to a unique point on the so-called *real number line* also known as *real number axis*:



The four fundamental arithmetic operations are:

Operation	Examples
Addition	$12+(-3)=9, \quad (-7)+5=-2, \quad (-12)+(-13)=-25$
Subtraction	$13-4=9, \quad -2-(-3)=1, \quad (-3)-4=-7$
Multiplication	$12\cdot 3=36, \quad 3\cdot(-5)=-15, \quad (-8)\cdot(-4)=32$
Division	$44\div 4=11, \quad \dfrac{-18}{3}=-6, \quad \dfrac{27}{-3}=-9, \quad \dfrac{-12}{-4}=3$

The most important properties of the real numbers $\mathbb{R}$ for the two operations $+$ and $\cdot$ (or $\times$) are listed below. The numbers x, y and z can be any real numbers.

Property	Addition	Multiplication
Identity	$x+0=x$	$x\cdot 1=x$
Inverse	$x+(-x)=0$	$x\cdot\dfrac{1}{x}=1,\ (x\neq 0)$
Commutative	$x+y=y+x$	$x\cdot y=y\cdot x$
Associative	$x+(y+z)=(x+y)+z$	$x\cdot(y\cdot z)=(x\cdot y)\cdot z$
Distributive	$x\cdot(y+z)=x\cdot y+x\cdot z$	

We would also like to recall that we have $x\cdot 0=0$ and $\dfrac{0}{x}=0$ for any $x\neq 0$, whereas $\dfrac{x}{0}$ is *undefined!*

The order of operations agreement:

To prevent of any ambiguities the following order of operations agreement have been established:

Step 1. Perform operations inside grouping symbols such as ( ), [ ], { } or fraction bar;

Step 2. Simplify exponential expressions (We will review exponentials in Section 0.3);

Step 3. Do multiplication and division as they occur from left to right;

Step 4. Do addition and subtraction as they occur from left to right.

Exercises

1. Evaluate the following expressions:

   (a) $24 - 13 + 12 - 17$

   (b) $21 - (-14) - 43 - 12$

   (c) $-18 - 49 - (18 - 77)$

   (d) $-37 + (-12) - (-13) + 17$

   (e) $-25 + 2(-31 + 1) - 3(3 - 4)$

   (f) $17 - (12 + 5) + 4(11 - (-11))$

   (g) $(12 - 3(12 - 3))(3 - 2(3 - 12))$

   (h) $\dfrac{-3 + 3}{4 - (-3 + 1)} \div -2(3 - 7)$

   (i) $13 - \dfrac{-12 + 3}{-3} + (-12)[-1 + 9 - 8]$

   (j) $12 - 24(8 - 5) \div 4$

   (k) $-4[16 - (7 - 1)] \div 10$

   (l) $6 \div [4 - (6 - 8)] - 2(3 + 13)$

   (m) $-27 \div 9 - 4(15) - (-13 - 11)$

   (n) $24 \div \dfrac{3 - 12}{8 - 5} - (-5)$

   (o) $7 - 6[1 - (2 - (-3))] \div -3$

   (p) $\dfrac{-19 + (-2)}{-29 + 36} \div (12 \div (-4))$

   (q) $\dfrac{1 - 13}{-2 - 4} \div \dfrac{-15 + 1}{-3 + 10}$

0.2 Operations on Rational Numbers

A rational number is a number that can be written in the form of $\frac{a}{b}$, where a and b are integers and $b \neq 0$. The numbers $\frac{-2}{5}$, $\frac{135}{7}$ and $0.35 = \frac{35}{100}$ are a few examples of rational numbers. There are infinitely many real numbers which are not rational numbers, for instance $\sqrt{2}$, $\frac{\sqrt{5}}{12}$ and π.

It is easily seen that any integer is a rational number with denominator 1. Two fractions are *equivalent* if one of them can be obtained from the other one by eliminating the common factors from numerator and denominator. For instance the two fractions $\frac{18}{30}$ and $\frac{3}{5}$ are equivalent because $\frac{18}{30} = \frac{\cancel{2} \cdot \cancel{3} \cdot 3}{\cancel{2} \cdot \cancel{3} \cdot 5} = \frac{3}{5}$

Multiplication and division of rational numbers are defined as follows:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

The division of fractions can also be written by fraction bar as follows:

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

To add or subtract fractions with different denominators, first we need to find the *least common multiple* (**L.C.M.**) of the denominators which is called the **least common denominator**. After finding the least common denominator we will rewrite the fractions as equivalent fractions with a common denominator and finally we follow the following simple rules:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

Exercises

1. Evaluate and simplify:

   (a) $\dfrac{-5}{12} + \dfrac{2}{3}$

   (b) $-\dfrac{7}{15} + \dfrac{13}{5}$

   (c) $\dfrac{5}{12} + \dfrac{3}{-8}$

   (d) $\dfrac{5}{8} - \dfrac{11}{12}$

   (e) $\dfrac{4}{5} - \dfrac{-5}{12}$

   (f) $\dfrac{-3}{4} - (-\dfrac{5}{6})$

   (g) $\dfrac{7}{16} + \dfrac{-3}{4} - \dfrac{5}{8}$

   (h) $-\dfrac{1}{8} - \dfrac{-17}{12} - \dfrac{1}{3}$

   (i) $\dfrac{5}{18} - \dfrac{-5}{6} + \dfrac{2}{9}$

   (j) $\dfrac{7}{12} - \dfrac{3}{4}(-\dfrac{1}{2})$

   (k) $(\dfrac{-2}{3})(\dfrac{-9}{8}) + (\dfrac{5}{8})(\dfrac{-1}{25})$

   (l) $(\dfrac{-2}{3} - \dfrac{1}{6})(\dfrac{-9}{2} + \dfrac{5}{3})$

   (m) $\dfrac{-7}{12} \div \dfrac{3}{4} - \dfrac{1}{8}$

   (n) $\dfrac{-7}{8} - \dfrac{16}{3} \div \dfrac{8}{9}$

   (o) $\dfrac{-5}{8} \div \dfrac{15}{16}(\dfrac{3}{2})$

   (p) $(\dfrac{5}{2} - \dfrac{4}{5}) \div (\dfrac{2}{5} + \dfrac{5}{4})$

   (q) $(\dfrac{-5}{3} + 1)(1 - \dfrac{3}{2})$

   (r) $(\dfrac{5}{16} + \dfrac{-7}{40})(\dfrac{-4}{11} + \dfrac{3}{2})$

0.3 Exponents

Repeated multiplication of the same factor can be written using an exponent.

Definition 0.1 *a^n (read[1] "a to the n-th power") is an exponential expression, where $n > 0$ is an integer, defined as*

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}.$$

In this exponential expression a is called the **base** *and n is the* **exponent** *(or* **power***).*

Note, in particular, that $a^1 = a$. We now extent this definition, as follows, to also include the non-positive exponents:

$$a^0 = 1 \quad (a \neq 0),$$
$$a^{-n} = \frac{1}{a^n} \quad (n > 0,\ a \neq 0).$$

Examples

Evaluate:

1. $-12^1 = -12$
2. $4^3 = 4 \cdot 4 \cdot 4 = 64$
3. $-5^4 = -5 \cdot 5 \cdot 5 \cdot 5 = -625$
4. $(-5)^4 = (-5) \cdot (-5) \cdot (-5) \cdot (-5) = 625$
5. $(-2)^5 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) = -32$
6. $(1345)^0 = 1$
7. $(12)^{-1} = \dfrac{1}{(12)^1} = \dfrac{1}{12}$
8. $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$
9. $(-3)^{-3} = \dfrac{1}{(-3)^3} = \dfrac{1}{-27}$
10. $(-4)^{-2} = \dfrac{1}{(-4)^2} = \dfrac{1}{16}$

[1] a^2 is also read "a squared", and a^3 is also read "a cubed".

Exercises

1. Evaluate the following expressions:

   (a) $24 \div 2^3 - 12 \div 2^2$

   (b) $27 \div (5-2)^2 + (-3)^2 \cdot 4$

   (c) $(-3)^2 + 12(-3^2 + 7)$

   (d) $(-2)^4 \cdot 3^3 - (162)^1 + 5^0$

   (e) $16 - 3(8-3)^2 \div 5$

   (f) $135^0 - 0^{135} + (-2 + 3^1)^{-1}$

   (g) $(8 - 3^2)^{102} - (2^3 - 9)^{101}$

   (h) $18 \div 3 - 2^3 \cdot 5 - 3^2$

   (i) $18 \div (9 - 2^3) + (-3^{-2} + (-2)^3)$

   (j) $7 - [3 - (1-3)^2]^2$

   (k) $-4 \cdot 2^3 - \dfrac{1-13}{2^2 \cdot 3}$

   (l) $\dfrac{3 \cdot 2^5}{2^3(4^2 - 1)} \cdot \dfrac{(-5)^2}{-5^2}$

   (m) $\dfrac{(-4)^2(-2)^3}{2^5} \div \dfrac{3^{-1}}{(-3)^2}$

   (n) $\dfrac{-12 - 2^2}{-10 + (-2)^2} - \dfrac{2 - 3^0}{-3^0 + 5}$

   (o) $12^{-1} + \dfrac{-2^2}{5 - 2^1} - \dfrac{2^{-2}}{-2^2}$

   (p) $\dfrac{4 \cdot 3^2}{-2^3 . (-3)} - \dfrac{3^3 - 7}{2^3 + 2} + \dfrac{4^2 - 2^4}{2^3}$

   (q) $\dfrac{-11^0 + 2^4}{3^2 - 2^3} \div \dfrac{4^1 - 3^2}{-3^2 - 1}$

   (r) $\dfrac{-3^2 \cdot 2^4}{4^2 \cdot 3} + \dfrac{7^0 - 0^7}{(-2)^2 \div 2^{-2}}$

0.4 Final Answers to Exercises

Section 0.1

1. (a) 6
   (b) -20
   (c) -8
   (d) -19
   (e) -82
   (f) 88
   (g) -315
   (h) 0
   (i) 10
   (j) -6
   (k) -4
   (l) -31
   (m) -39
   (n) -3
   (o) -1
   (p) 1
   (q) -1

Section 0.2

1. (a) $\frac{1}{4}$
   (b) $\frac{32}{15}$
   (c) $\frac{1}{24}$
   (d) $\frac{-7}{24}$
   (e) $\frac{73}{60}$
   (f) $\frac{1}{12}$
   (g) $\frac{-15}{16}$
   (h) $\frac{23}{24}$
   (i) $\frac{4}{3}$
   (j) $\frac{23}{24}$
   (k) $\frac{29}{40}$
   (l) $\frac{85}{36}$
   (m) $\frac{-65}{72}$
   (n) $\frac{-55}{8}$
   (o) $\frac{-4}{9}$
   (p) $\frac{34}{33}$
   (q) $\frac{1}{3}$
   (r) $\frac{5}{32}$

Section 0.3

1. (a) 0
   (b) 39
   (c) -15
   (d) 271
   (e) 1
   (f) 2
   (g) 2
   (h) -43
   (i) $\frac{89}{9}$
   (j) 6
   (k) -31
   (l) $\frac{-4}{5}$
   (m) -108
   (n) $\frac{29}{12}$
   (o) $\frac{-19}{16}$
   (p) $\frac{-1}{2}$
   (q) 30
   (r) $\frac{-47}{16}$

Chapter 1

Operations on Algebraic Expressions

In mathematics, **an algebraic expression is a meaningful expression built up from numbers, variables, and the algebraic operations.**

In this chapter, we will introduce algebraic operations on algebraic expressions, similar to those described in previous chapter. Let us start with the definition of **integer exponents in algebra**.

1.1 Integer Exponents in Algebra

Definition 1.1 *a^n (read[1] "a to the n-th power") is an exponential expression, where a is the* **base** *and the integer $n > 0$ is the* **exponent** *(or* **power***), defined as*

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}.$$

Note, in particular, that $a^1 = a$. We now extent this definition, as follows, to also include the non-positive exponents:

$$a^0 = 1, \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n},$$

[1] a^2 is also read "a squared", and a^3 is also read "a cubed".

where $a \neq 0$.

The Main Properties

- $a^m \cdot a^n = a^{m+n}$;
- $\dfrac{a^m}{a^n} = a^{m-n}$;
- $(a^m)^n = a^{mn}$;
- $(ab)^m = a^m b^m$;
- $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$;
- $\dfrac{a^{-m}}{b^{-n}} = \dfrac{b^n}{a^m}$;
- $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$.

Examples

Simplify and express the answer with positive exponents only:

(1)

$$\frac{a^2 \cdot a^3}{(a^2)^3} = \frac{a^{2+3}}{a^{2\cdot 3}} = \frac{a^5}{a^6} = a^{5-6} = a^{-1} = \frac{1}{a^1} = \frac{1}{a}.$$

(2)

$$\left(\frac{2a^2}{3b^3}\right)^2 = \frac{(2a^2)^2}{(3b^3)^2} = \frac{2^2(a^2)^2}{3^2(b^3)^2} = \frac{4a^4}{9b^6}.$$

(3)

$$\frac{(3x^2)^{-2}}{(2y^3)^{-3}} = \frac{(2y^3)^3}{(3x^2)^2} = \frac{2^3(y^3)^3}{3^2(x^2)^2} = \frac{8y^9}{9x^4}.$$

(4)

$$\left(\frac{2y^3}{x^{-2}}\right)^4 = \frac{(2y^3)^4}{(x^{-2})^4} = \frac{16y^{12}}{x^{-8}} = 16x^8y^{12}.$$

(5)

$$\frac{(2xy)^{-3}(yx^{-1})^0}{((-x)^2y)^{-2}} = \frac{(x^2y)^2}{(2xy)^3} = \frac{x^4y^2}{8x^3y^3} = \frac{x}{8y}.$$

Exercises

1. Simplify and express the answer with positive exponents only:

(a) $\dfrac{a^2a^4}{(a^3)^2}$

(b) $\dfrac{x^3x^{-4}}{x^{11}}$

(c) $\dfrac{a^{14}a^{-2}}{(a^4)^3a^0}$

(d) $\dfrac{x^2y^6}{x^3y^5}$

(e) $\dfrac{x^2y(y^2)^3}{(x^3y^2)^{-1}}$

(f) $\left(\dfrac{x^{-1}y^2}{y^3x}\right)^3$

(g) $\dfrac{(a^3b^{-2})^{-3}(a^{-1}b^2)^0}{(a^4)^{-2}ab^5}$

(h) $\dfrac{(ab^0)^{-4}(a^2b^{-2})^{-3}}{(a^5b^{-3})^{-2}}$

(i) $\left(\dfrac{xy^{-3}}{y^2}\right)^0\left(\dfrac{-x^3y^2}{y^{-1}x}\right)^2$

(j) $\dfrac{(2x^3)^{-2}}{(-3x^4)^{-1}}$

(k) $\dfrac{(5x)^0y^{-2}}{(2x)^{-3}y^2}$

(l) $\dfrac{(-2)^{-2}x^{-1}y^2}{y^3x^{-2}}$

(m) $(-2a)^{-3}b^3(3a^2b^{-1})^2$

(n) $\left(\dfrac{x^{-2}}{-2y^3}\right)^3$

(o) $\left(\dfrac{2a^{-2}b^{-1}}{-a^{-1}b^3}\right)^{-2}$

(p) $\dfrac{(-2xy^{-3})^{-3}}{(3x^{-2}y^3)^{-2}}$

(q) $\dfrac{(x^2y^{-1})^4}{(y^{-5}x^{-2})^3}$

(r) $\dfrac{(a^2b^{-3})^2}{(b^{-2}a^{-3})^{-2}}$

(s) $\left(\dfrac{2r^{-3}s^2}{-4s^{-1}r^2}\right)^{-2}$

(t) $\dfrac{(x^{-2}y^3)^{-2}(6x^2y^5)^{-2}}{(-3xy^2)^{-1}}$

(u) $\dfrac{(2x^{-2}y^3)^{-1}(x^3y^{-2})^0}{(-x)^{-3}y^{-2}}$

(v) $\dfrac{(-4x^{-3}y^2z^{-1})^{-2}}{((-2x)^3y^{-2}z^2)^{-3}}$

(w) $\dfrac{(-2a^3b^2)^2(-5ab)^2}{a^{-1}(-b)^4(-ab^{-2})^{-3}}$

1.2 Polynomials and Their Operations

In this section we define a very simple form of algebraic expressions called **polynomials**.

Definition 1.2 *A* **polynomial** *in a variable, say x, is an expression in x whose terms may be arranged in descending powers of x as follows*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

*where n is a positive integer; $a_n, a_{n-1}, ..., a_1$ and a_0 (called the "***coefficients***" of $P(x)$) are real numbers (positive, negative or even zero) and where the variable x may assume any value. The highest power of x appearing in the expression is called the* **degree** *of the polynomial:*

$$\deg P(x) = n \quad \text{provided} \quad a_n \neq 0.$$

And finally, the term $a_n x^n$ is called the **leading term** *of $P(x)$.*

Notes: (1) A polynomial with one term only is often referred to as a *monomial*, a polynomial with two terms as *binomial*, and a polynomial with three terms as *trinomial*;

(2) A polynomial may contain more than one variable in it, for instance $P(x, y) = 2x^3y + 5xy - y^2$ is a polynomial in two variables and $Q(a, b, c) = a^2 + b^2 + c^2 - ab - bc - ca$ is a polynomial in three variables;

(3) If two monomials have the same variable(s) and the corresponding powers of all the variables involved are equal, then the two monomials are said to be "**similar terms**" (or "**like terms**"). For example, $4x^2yz^3$ and $-3x^2yz^3$ are like terms, whereas $5xyz^2$ and $6xy^2z^2$ are not like terms.

Arithmetic Operations on Polynomials

(I) Addition/Subtraction: To add or subtract two polynomials we simply add or subtract the coefficients of the like terms. Here are some examples:
(1)

$$\begin{aligned}(2x^2 - 3x + 5) + (x^2 + 10x - 2) &= (2+1)x^2 + (-3+10)x + (5 + (-2)) \\ &= 3x^2 + 7x + 3,\end{aligned}$$

(2)

$$(2x^3 + x^2 - 4x + 11) + (6x^3 - 6x + 1) = 8x^3 + x^2 - 10x + 12,$$

(3)

$$\begin{aligned}(5x^2 - x + 3) - (2x^2 - 7x + 6) &= (5-2)x^2 + ((-1) - (-7))x + (3-6)\\ &= 3x^2 + 6x - 3.\end{aligned}$$

(II) Multiplication: To multiply two polynomials we multiply each term of the first polynomial to each term of the second polynomial, based on the *Distribution Law*:

$$\begin{aligned}A(K + L + \cdots) &= AK + AL + \cdots && (\bigstar),\\ (A + B + \cdots)(K + L + \cdots) &= AK + AL + \cdots + BK + BL + \cdots && (\bigstar\bigstar).\end{aligned}$$

Here are some examples:

(4)

$$3x^2(x+5) = 3x^2 \cdot x + 3x^2 \cdot 5 = 3x^3 + 15x^2,$$

(5)

$$(x^2+1)(2x-7) = x^2 \cdot 2x + x^2 \cdot (-7) + 1 \cdot 2x + 1 \cdot (-7) = 2x^3 - 7x^2 + 2x - 7,$$

(6)

$$(a+b)(a^2 - ab + b^2) = a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 = a^3 + b^3.$$

We shall now gather some of the very important identities with their associated nicknames which will be in constant use throughout the course. It should be emphasized, however, that the following list is by no means complete.

- "**square of a binomial**" identity:

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{and} \qquad (a-b)^2 = a^2 - 2ab + b^2}$$

- "**difference of squares**" identity[2]:

$$\boxed{(a-b)(a+b) = a^2 - b^2}$$

[2] Also known as the "conjugate" identity.

- "**sum/difference of cubes**" identity:

$$\boxed{(a+b)(a^2-ab+b^2)=a^3+b^3 \qquad \text{and} \qquad (a-b)(a^2+ab+b^2)=a^3-b^3}$$

(III) Division: The division of two polynomials entitled "Long Division" is not covered in this course; we only deal with the division of a polynomial by a monomial.

Examples

(7)

$$\frac{4x^3+6x^2-x}{2x}=\frac{4x^3}{2x}+\frac{6x^2}{2x}-\frac{x}{2x}=2x^2+3x-\frac{1}{2},$$

(8)

$$\frac{7a^2b^3-8a^3b+12ab^4}{4ab}=\frac{7a^2b^3}{4ab}-\frac{8a^3b}{4ab}+\frac{12ab^4}{4ab}=\frac{7}{4}ab^2-2a^2+3b^3,$$

Exercises

1. Expand and simplify completely.

   (a) $(4x^3 - 2x + 11) + (x^2 - 3x - 10)$

   (b) $(2a^3 - a^2 + 4a - 5) - (5a + a^3 - 2a^2 - 5)$

   (c) $(3x^2y - 4xy + 4y^2) - 2(xy^2 - 2xy + 2y^2)$

   (d) $4(x^2 - 2x + 3) + 2(1 - x)$

   (e) $-3(a^4 - 2a^3 + a - 12) - (6a^3 - 3a^4 + 2a^2 - a - 24)$

   (f) $4(x^2 - x + 1) + 2(x^3 + 7x - 3) - 3(2x^3 - x^2 + x - 1)$

   (g) $-2(1 - b^3) - (b^3 + 2b^2 + 3b - 1) - (b^4 + 2b - 1)$

   (h) $2(2x^2 - 3y + xy) - 3(4y - 3x^2 - 4xy)$

   (i) $-3(x^2 + x^2y - 3xy^2) + 2(xy + 2x^2y - xy^2 + 3x^2) - (xy^2 - yx^2)$

   (j) $2a(a^2 - 3ab + 2b) - 3b(2a^2 - 4a + 1)$

   (k) $5t^2(t^3 - 2t^2 + t - 1) + 3t(t^4 - t^2 + 4)$

   (l) $3xy^2(x - y) + 2x(xy^2 - x + y^3)$

   (m) $(3a - 5)(a + 7)$

   (n) $(7x - 3)(2x - 2)$

   (o) $(2x^2 - 3)(x^2 + 1)$

   (p) $(x - 2y)(3x + y)$

   (q) $(2s^2 - 3t)(s + t^2)$

   (r) $2x(x - 3)(3x + 2)$

   (s) $3x^2(2x^2 + y)(x - y)$

   (t) $(2u + 3)(u^2 - 2u + 1)$

(u) $(x-1)(x^3-x^2+7)$

(v) $(2y-3)(y^2-3y+1)$

(w) $(3x^2-2x+5)(x^2-2)$

(x) $(3x-y)(x+2xy+y)$

(y) $2x(3x+2y)(2x+2y+3)$

2. Use the special identities to expand and simplify the following expressions.

(a) $(x+3)^2$

(b) $(2x+1)^2$

(c) $(4x-1)^2$

(d) $(2x-3y)^2$

(e) $(1-3x)^2$

(f) $(5x+3)^2$

(g) $(-4x-3)^2$

(h) $(-2x-5y)^2$

(i) $(x-2)(x+2)$

(j) $(2x-1)(2x+1)$

(k) $(3x-2y)(3x+2y)$

(l) $(5x^2+2)(5x^2-2)$

(m) $(4x+3)(4x-3)$

(n) $(3x^2-2y)(3x^2+2y)$

(o) $(y+2)(y^2-2y+4)$

(p) $(x-3)(x^2+3x+9)$

(q) $(2x-1)(4x^2+2x+1)$

(r) $(3x+2)(9x^2-6x+4)$

(s) $(5x-y)(25x^2+5xy+y^2)$

(t) $(2x+3y)(4x^2-6xy+9y^2)$

3. Expand and simplify. Use the identities if it is applicable.

(a) $(x-1)(x^2+2x+1)$

(b) $(x-1)^2(x+1)^2$

(c) $(2y-1)(4y^2-y+1)$

(d) $(2x-y)(4x^2+2xy+y^2)$

(e) $3(a+2)^2-(2a+1)(2a-1)$

(f) $(x-1)^3$

(g) $(1+2x)^3$

(h) $-2x(2x+1)(4x^2-2x+1)$

(i) $(x^2+y^2)(x+y)(x-y)$

(j) $3(x^2-1)(x^2+1)-(x^2+2)^2$

1.3 Roots and Radicals

Definition 1.3 *The* n^{th} *root of* A *is a number whose* n^{th} *power is* A*, that is to say,*

$$\sqrt[n]{A} = r \qquad \text{if} \quad r^n = A,$$

where $n \geq 1$ *is an integer. In this notation,* n *is called the "***index***" and* A *is called the "***radicand***" of the radical. In the absence of index, the index is understood as* 2*, that is to say,*

$$\sqrt{A} = \sqrt[2]{A}.$$

Remark 1. If n is odd (i.e., if $n = 1, 3, 5, ...$), then such r always exists; it is unique and has the same sign as A does. And if n is even (if $n = 2, 4, 6, ...$), then r exists if and only if $A \geq 0$.

Remark 2. For any real number a, $\sqrt{a^2} = |a|$ (read absolute value of a) that is defined as $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$. Therefore $\sqrt{2^2} = \sqrt{4} = 2$ and $\sqrt{(-2)^2} = \sqrt{4} = 2$.

Examples

(1) We have $\sqrt{25} = 5$, as $5^2 = 25$ and $5 > 0$.

(2) We have $\sqrt[3]{-8} = -2$ as $(-2)^3 = -8$.

(3) We have $-\sqrt[4]{81} = -3$; in this example note that we gave the answer as -3 not because $(-3)^4 = 81$, but because there already exists a minus sign before the radical. In fact without that minus sign, the answer would have been 3.

(4) Assuming all the variables are positive, we have $\sqrt{x^6} = x^3$ because $(x^3)^2 = x^6$; and also $\sqrt[4]{y^8} = y^2$ since $(y^2)^4 = y^8$

Multiplication, Division and Power Rules of Radicals:

- $\sqrt[n]{A}\sqrt[n]{B} = \sqrt[n]{AB}$;
- $\sqrt[n]{\dfrac{A}{B}} = \dfrac{\sqrt[n]{A}}{\sqrt[n]{B}}$;
- $\sqrt[n]{A} = \sqrt[mn]{A^m}$.

Examples

(5) $\sqrt{2}\sqrt{8} = \sqrt{16} = 4;$

(6) $\sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2};$

(7) $\sqrt{12x^2} = \sqrt{4x^2}\sqrt{3} = 2x\sqrt{3}$, provided that $x > 0$;

(8) $\sqrt[3]{96} = \sqrt[3]{8}\sqrt[3]{12} = 2\sqrt[3]{12};$

(9) $\dfrac{\sqrt{14}}{\sqrt{7}} = \sqrt{\dfrac{14}{7}} = \sqrt{2};$

(10) $\sqrt{\dfrac{5}{16} + \dfrac{1}{4}} = \sqrt{\dfrac{9}{16}} = \dfrac{3}{4};$

Addition and Subtraction of Radicals

Expressions containing radicals can be added or subtracted if they are **similar**. Two radical expressions are called similar if they have the same radicand and the same index.

Examples

(11) $\sqrt{27} + 3\sqrt{2} + \sqrt{12} - 2\sqrt{18} = 3\sqrt{3} + 3\sqrt{2} + 2\sqrt{3} - 6\sqrt{2} = 5\sqrt{3} - 3\sqrt{2}$

(12) $(3+\sqrt{6})(2-\sqrt{6}) = 3\cdot 2 - 3\sqrt{6} + 2\sqrt{6} - \sqrt{6}\sqrt{6} = 6 - \sqrt{6} - 6 = -\sqrt{6};$

Rationalizing Denominators

Given a fraction with radical(s) in its denominator, sometimes we need to remove the radical(s) from the denominator without changing the value of the whole fraction, a proces known as **"rationalizing the denominator"**. In order to do so, one has to multiply the denominator as well as the numerator by a suitable expression, so that the radical(s) disappear from the denominator. We illustrate this through a number of examples.

Examples

Rationalize the denominator and simplify if possible:

(13) $\dfrac{2}{\sqrt{5}} = \dfrac{2}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5};$

(14) $\dfrac{8}{\sqrt{11}-3} = \dfrac{8}{\sqrt{11}-3} \cdot \dfrac{\sqrt{11}+3}{\sqrt{11}+3} = \dfrac{8(\sqrt{11}+3)}{\left(\sqrt{11}\right)^2 - 3^2} = \dfrac{8(\sqrt{11}+3)}{2} = 4(\sqrt{11}+3);$

(15) $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} \cdot \dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \dfrac{5-2\sqrt{15}+3}{5-3} = \dfrac{8-2\sqrt{15}}{2} = 4-\sqrt{15}.$

Definition 1.4 *The rational exponents are defined as follows:*

$$A^{\frac{m}{n}} = \sqrt[n]{A^m} = \left(\sqrt[n]{A}\right)^m .$$

In particular, we have $A^{\frac{1}{n}} = \sqrt[n]{A}$*; in other words,* $A^{\frac{1}{n}}$ *is just another notation for the* n^{th} *root of* A*.*

Examples

(16) $4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8;$

(17) $(-3)^{5/3} \cdot (-3)^{4/3} = (-3)^{5/3+4/3} = (-3)^{9/3} = (-3)^3 = -27;$

(18) $\left(8x^9y^6\right)^{1/3} = 8^{1/3}(x^9)^{1/3}(y^6)^{1/3} = 2x^3y^2;$

(19) $2^{3/2} - \sqrt{50} = \sqrt{8} - 5\sqrt{2} = 2\sqrt{2} - 5\sqrt{2} = -3\sqrt{2};$

(20)

$$\begin{aligned}
\frac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{48}-\sqrt{32}+\sqrt{50}} &= \frac{3+\sqrt{6}}{5\sqrt{3}-8\sqrt{3}-4\sqrt{2}+5\sqrt{2}} \\
&= \frac{3+\sqrt{6}}{\sqrt{2}-3\sqrt{3}} \\
&= \frac{(3+\sqrt{6})(\sqrt{2}+3\sqrt{3})}{(\sqrt{2}-3\sqrt{3})(\sqrt{2}+3\sqrt{3})} \\
&= \frac{12\sqrt{2}+11\sqrt{3}}{-25}.
\end{aligned}$$

Exercises

1. Simplify.

   (a) $\sqrt{\dfrac{25}{36}}$

   (b) $\sqrt[3]{-125}$

   (c) $\sqrt{\dfrac{16}{81}}$

   (d) $\sqrt[4]{\dfrac{16}{81}}$

   (e) $-\sqrt{\dfrac{36}{121}}$

   (f) $\sqrt[3]{\dfrac{-343}{27}}$

   (g) $-\sqrt{16}$

   (h) $-\sqrt[5]{32}$

   (i) $\sqrt{-25}$

   (j) $\sqrt{18}$

   (k) $-\sqrt{12}$

   (l) $\sqrt{40}$

   (m) $\dfrac{\sqrt{180}}{3}$

   (n) $\dfrac{\sqrt{28}}{6}$

   (o) $\sqrt{300}$

   (p) $3\sqrt{50}$

   (q) $\dfrac{-2\sqrt{45}}{9}$

   (r) $17\sqrt{80}$

   (s) $-3\sqrt{121}$

   (t) $-8\sqrt{48}$

   (u) $7\sqrt{288}$

2. Simplify.

   (a) $\sqrt{8} - \sqrt{2}$

   (b) $\sqrt{12} + \sqrt{27}$

   (c) $7\sqrt{2} - \sqrt{20} + 2\sqrt{18}$

   (d) $7\sqrt{80} - 3\sqrt{50} - 2\sqrt{5}$

   (e) $4\sqrt{75} + 3\sqrt{48} - \sqrt{12}$

   (f) $\sqrt{32} + \sqrt{45} - \sqrt{98} - 7\sqrt{5}$

   (g) $\sqrt{125} - 2\sqrt{27} - 3\sqrt{5} + 3\sqrt{12}$

   (h) $4\sqrt{18} - 3\sqrt{64} + 2\sqrt{50} + 7\sqrt{25}$

   (i) $3(2\sqrt{12} - \sqrt{99})$

   (j) $-7(3\sqrt{8} - 2\sqrt{45})$

   (k) $\sqrt{2}(-2\sqrt{32} + 5\sqrt{18})$

   (l) $2\sqrt{3}(3 - \sqrt{3})$

(m) $3\sqrt{3}(2\sqrt{75}-7\sqrt{27})$

(n) $\sqrt{18}(1+2\sqrt{2})$

(o) $2\sqrt{3}(3\sqrt{8}-2\sqrt{3})$

(p) $\sqrt{12}(3\sqrt{6}+\sqrt{10})$

(q) $(\sqrt{2}-3)(\sqrt{2}+3)$

(r) $(2\sqrt{3}+1)(2\sqrt{3}-1)$

(s) $(4\sqrt{2}-5)(4\sqrt{2}+5)$

(t) $(3+2\sqrt{5})(2-\sqrt{5})$

(u) $(\sqrt{8}-2\sqrt{3})(2\sqrt{20}-3)$

(v) $(3\sqrt{3}-\sqrt{18})(3\sqrt{2}-\sqrt{27})$

(w) $(1+\sqrt{2})^2$

(x) $(\sqrt{3}-3\sqrt{2})^2$

(y) $(\sqrt{5}-2\sqrt{6})^2$

(z) $(2\sqrt{3}-3\sqrt{2})^2$

3. Simplify, assuming that all the variables are positive.

(a) $\sqrt{x^8}$

(b) $\sqrt{x^5}$

(c) $\sqrt{16y^{16}}$

(d) $\sqrt{x^7}$

(e) $\sqrt{a^2b^3}$

(f) $\sqrt{18x^4}$

(g) $\dfrac{\sqrt{8x^9}}{\sqrt{2x^3}}$

(h) $\sqrt{3x^5}\sqrt{15x^3}$

(i) $\sqrt{3a^3b^7}\sqrt{27ab^3}$

(j) $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$

(k) $\sqrt{6x}(\sqrt{3x^3}-\sqrt{2})$

(l) $\sqrt{15ab}(\sqrt{5a}-\sqrt{3b})$

(m) $(\sqrt{3x}+\sqrt{2x^3y})(\sqrt{18x}+\sqrt{12y})$

(n) $(3\sqrt{x}+\sqrt{2y})^2$

4. Rationalize the denominator.

(a) $\dfrac{2}{\sqrt{2}}$

(b) $\dfrac{-2}{\sqrt{3}}$

(c) $\dfrac{6}{\sqrt{10}}$

(d) $\dfrac{15}{\sqrt{5}}$

(e) $\dfrac{-15\sqrt{3}}{\sqrt{5}}$

(f) $\dfrac{1+\sqrt{3}}{\sqrt{3}}$

(g) $\dfrac{3}{1+\sqrt{2}}$

(h) $\dfrac{8}{5-\sqrt{3}}$

(i) $\dfrac{\sqrt{12}}{\sqrt{7}-\sqrt{3}}$

(j) $\dfrac{7}{2\sqrt{2}+1}$

(k) $\dfrac{26}{5-2\sqrt{3}}$

(l) $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

(m) $\dfrac{7+\sqrt{3}}{3-\sqrt{3}}$

(n) $\dfrac{\sqrt{8}-6}{\sqrt{8}+6}$

(o) $\dfrac{3\sqrt{2}-\sqrt{3}}{2\sqrt{2}+\sqrt{3}}$

(p) $\dfrac{2\sqrt{5}-\sqrt{15}}{\sqrt{15}-3\sqrt{3}}$

5. Simplify. You may assume x and y are positive.

(a) $(-27)^{\frac{2}{3}}$

(b) $16^{\frac{5}{4}}$

(c) $-25^{\frac{3}{2}}$

(d) $(-8)^{\frac{5}{3}}$

(e) $8^{\frac{-5}{3}}$

(f) $(-25)^{-\frac{1}{2}}$

(g) $(\dfrac{4}{9})^{\frac{3}{2}}$

(h) $(-2)^{\frac{2}{3}}(-2)^{\frac{1}{3}}$

(i) $4^{\frac{-2}{3}}4^{\frac{8}{3}}$

(j) $\dfrac{7^{\frac{2}{3}}}{7^{\frac{-4}{3}}}$

(k) $\dfrac{6^{\frac{2}{5}}6^{\frac{1}{5}}}{6^{\frac{3}{5}}}$

(l) $3^{\frac{5}{2}}-\sqrt{48}$

(m) $(81x^4y^8)^{\frac{3}{4}}$

(n) $\left(\dfrac{8x^3}{27y^6}\right)^{\frac{-1}{3}}$

(o) $\dfrac{(8x)^{\frac{3}{2}}(y)^{\frac{5}{2}}}{\sqrt{2xy}}$

(p) $\dfrac{\sqrt[3]{81x^2y}(xy^2)^{\frac{4}{3}}}{x\sqrt[3]{3xy}}$

1.4 Final Answers to Exercises

Section 1.1

1. (a) 1
   (b) $\dfrac{1}{x^{12}}$
   (c) 1
   (d) $\dfrac{y}{x}$
   (e) x^5y^9
   (f) $\dfrac{1}{x^6y^3}$
   (g) $\dfrac{b}{a^2}$
   (h) 1
   (i) x^4y^6
   (j) $\dfrac{-3}{4x^2}$
   (k) $\dfrac{8x^3}{y^4}$
   (l) $\dfrac{x}{4y}$
   (m) $\dfrac{9ab}{-8}$
   (n) $\dfrac{1}{-8x^6y^9}$
   (o) $\dfrac{a^2b^8}{4}$
   (p) $\dfrac{-9y^{15}}{8x^7}$
   (q) $x^{14}y^{11}$
   (r) $\dfrac{1}{a^2b^{10}}$
   (s) $\dfrac{4r^{10}}{s^6}$
   (t) $\dfrac{-x}{12y^{14}}$
   (u) $\dfrac{-x^5}{2y}$
   (v) $\dfrac{-32x^{15}z^8}{y^{10}}$
   (w) $\dfrac{-100a^{12}}{b^4}$

Section 1.2

1. (a) $4x^3 + x^2 - 5x + 1$
   (b) $a^3 + a^2 - a$
   (c) $3x^2y - 2xy^2$
   (d) $4x^2 - 10x + 14$
   (e) $-2a^2 - 2a + 60$
   (f) $-4x^3 + 7x^2 + 7x + 1$
   (g) $-b^4 + b^3 - 2b^2 - 5b$
   (h) $13x^2 + 14xy - 18y$
   (i) $3x^2 + 2x^2y + 6xy^2 + 2xy$
   (j) $2a^3 - 12a^2b + 16ab - 3b$
   (k) $8t^5 - 10t^4 + 2t^3 - 5t^2 + 12t$
   (l) $5x^2y^2 - xy^3 - 2x^2$
   (m) $3a^2 + 16a - 35$
   (n) $14x^2 - 20x + 6$
   (o) $2x^4 - x^2 - 3$
   (p) $3x^2 - 5xy - 2y^2$

(q) $2s^3 + 2s^2t^2 - 3st - 3t^3$

(r) $6x^3 - 14x^2 - 12x$

(s) $6x^5 - 6x^4y + 3x^3y - 3x^2y^2$

(t) $2u^3 - u^2 - 4u + 3$

(u) $x^4 - 2x^3 + x^2 + 7x - 7$

(v) $2y^3 - 9y^2 + 11y - 3$

(w) $3x^4 - 2x^3 - x^2 + 4x - 10$

(x) $3x^2 + 6x^2y + 2xy - 2xy^2 - y^2$

(y) $12x^3 + 20x^2y + 18x^2 + 8xy^2 + 12xy$

2. (a) $x^2 + 6x + 9$

(b) $4x^2 + 4x + 1$

(c) $16x^2 - 8x + 1$

(d) $4x^2 - 12xy + 9y^2$

(e) $1 - 6x + 9x^2$

(f) $25x^2 + 30x + 9$

(g) $16x^2 + 24x + 9$

(h) $4x^2 + 20xy + 25y^2$

(i) $x^2 - 4$

(j) $4x^2 - 1$

(k) $9x^2 - 4y^2$

(l) $25x^4 - 4$

(m) $16x^2 - 9$

(n) $9x^4 - 4y^2$

(o) $y^3 + 8$

(p) $x^3 - 27$

(q) $8x^3 - 1$

(r) $27x^3 + 8$

(s) $125x^3 - y^3$

(t) $8x^3 + 27y^3$

3. (a) $x^3 + x^2 - x - 1$

(b) $x^4 - 2x^2 + 1$

(c) $8y^3 - 6y^2 + 3y - 1$

(d) $8x^3 - y^3$

(e) $-a^2 + 12a + 13$

(f) $x^3 - 3x^2 + 3x - 1$

(g) $1 + 12x^2 + 6x + 8x^3$

(h) $-16x^4 - 2x$

(i) $x^4 - y^4$

(j) $2x^4 - 4x^2 - 7$

Section 1.3

1. (a) $\dfrac{5}{6}$

(b) -5

(c) $\dfrac{4}{9}$

(d) $\dfrac{2}{3}$

(e) $\dfrac{-6}{11}$

(f) $\dfrac{-7}{3}$

(g) -4

(h) -2

(i) undefined!

(j) $3\sqrt{2}$

(k) $-2\sqrt{3}$

(l) $2\sqrt{10}$

(m) $2\sqrt{5}$

(n) $\dfrac{\sqrt{7}}{3}$

(o) $10\sqrt{3}$

(p) $15\sqrt{2}$

(q) $\dfrac{-2\sqrt{5}}{3}$

(r) $68\sqrt{5}$

(s) -33

(t) $-32\sqrt{3}$

(u) $84\sqrt{2}$

2. (a) $\sqrt{2}$

(b) $5\sqrt{3}$

(c) $13\sqrt{2}-2\sqrt{5}$

(d) $26\sqrt{5}-15\sqrt{2}$

(e) $30\sqrt{3}$

(f) $-3\sqrt{2}-4\sqrt{5}$

(g) $2\sqrt{5}$

(h) $22\sqrt{2}+11$

(i) $12\sqrt{3}-9\sqrt{11}$

(j) $-42\sqrt{2}+42\sqrt{5}$

(k) 14

(l) $6\sqrt{3}-6$

(m) -99

(n) $3\sqrt{2}+12$

(o) $12\sqrt{6}-12$

(p) $18\sqrt{2}+2\sqrt{30}$

(q) -7

(r) 11

(s) 7

(t) $-4+\sqrt{5}$

(u) $8\sqrt{10}-6\sqrt{2}-8\sqrt{15}+6\sqrt{3}$

(v) $-45+18\sqrt{6}$

(w) $3+2\sqrt{2}$

(x) $21-6\sqrt{6}$

(y) $29-4\sqrt{30}$

(z) $30-12\sqrt{6}$

3. (a) x^4

(b) $x^2\sqrt{x}$

(c) $4y^8$

(d) $x^3\sqrt{x}$

(e) $ab\sqrt{b}$

(f) $3\sqrt{2}x^2$

(g) $2x^3$

(h) $3\sqrt{5}x^4$

(i) $9a^2b^5$

(j) $a-b$

(k) $3\sqrt{2}x^2 - 2\sqrt{3x}$

(l) $5a\sqrt{3b} - 3b\sqrt{5a}$

(m) $3x\sqrt{6} + 6\sqrt{xy} + 6x^2\sqrt{y} + 2xy\sqrt{6x}$

(n) $9x + 6\sqrt{2xy} + 2y$

4. (a) $\sqrt{2}$
   (b) $\dfrac{-2\sqrt{3}}{3}$
   (c) $\dfrac{3\sqrt{10}}{5}$
   (d) $3\sqrt{5}$
   (e) $-3\sqrt{15}$
   (f) $\dfrac{\sqrt{3}+3}{3}$
   (g) $3\sqrt{2} - 3$
   (h) $\dfrac{4(5+\sqrt{3})}{11}$
   (i) $\dfrac{\sqrt{21}+3}{2}$
   (j) $2\sqrt{2} - 1$
   (k) $2(5 + 2\sqrt{3})$
   (l) $5 + 2\sqrt{6}$
   (m) $\dfrac{12+5\sqrt{3}}{3}$
   (n) $\dfrac{11-6\sqrt{2}}{-7} = \dfrac{6\sqrt{2}-11}{7}$
   (o) $3 - \sqrt{6}$
   (p) $\dfrac{10\sqrt{3}+6\sqrt{15}-9\sqrt{5}-15}{-12}$

5. (a) 9
   (b) 32
   (c) -125
   (d) -32
   (e) $\dfrac{1}{32}$
   (f) undefined!
   (g) $\dfrac{8}{27}$
   (h) -2
   (i) 16
   (j) 49
   (k) 1
   (l) $5\sqrt{3}$
   (m) $27x^3y^6$
   (n) $\dfrac{3y^2}{2x}$
   (o) $16xy^2$
   (p) $3x^{2/3}y^{8/3}$

Chapter 2

Linear Equations and Formulae

An **equation** is an equality between two algebraic expressions and solving an equation means finding the value(s) for the unknowns in the equation that makes the equality true. We will start solving **linear equations** which are the simplest form of mathematical equations.

2.1 Solving Linear Equations

Definition 2.1 *Any polynomial equation which can be brought to the "standard" form*

$$ax + b = 0,$$

is called a **linear equation**. *As such, a linear equation has always one and only one solution*

$$x = -\frac{b}{a},$$

provided that $a \neq 0$.

Examples

(1) The equation $3x + 2 = 0$ is linear;

(2) The equation $3x^2 + 2x - 5 = 0$ is **not** linear; in fact it is an example of a quadratic equation which we will study afterwards;

(3) The equation $\frac{2x+6}{x+1} - 3 = 1$ is not linear as the left-hand side is not even a polynomial and indeed it contains fractions; note, however, that one may obtain a linear equation out of this after getting rid of the denominator.

How to Solve: In order to solve a linear equation one has to isolate the variable, say x, on one side of the equation and to have only a constant (a number) on the other side. This number, known as the **solution**, must satisfy the initial equation. In order to solve an equation we should follow the following cancelation rules:

$$A + C = B + C \iff A = B,$$

$$A - C = B - C \iff A = B.$$

These rules simply mean that one may send any term from one side to the other side by changing its sign. We also need to use the next two rules in order to isolate the variable:

$$A \cdot C = B \cdot C \iff A = B, \qquad C \neq 0,$$

$$\frac{A}{C} = \frac{B}{C} \iff A = B, \qquad C \neq 0.$$

After applying the first two rules and simplifying both sides of the equation, we often need to cancel the coefficient of the variable, x, by a simple division by that coefficient. This can be done by using the last two given rules. We shall now illustrate all this in a number of examples.

Examples

(4) To solve the equation $5x - 2(x+1) - 4 = -2x - 1$ for x, we proceed as follows:

$$\begin{aligned} 5x - 2(x+1) - 4 &= -2x - 1 \\ 5x - \cancel{2x} - 2 - 4 &= -\cancel{2x} - 1 \\ 5x = 6 - 1 = 5 \implies x &= 5/5 = 1. \end{aligned}$$

Thus, $x = 1$ is the solution of the original equation, as can easily be verified by the reader.

(5) To solve $4[2x-(x-2)] = -3(3-2x)$, we proceed as follows:

$$\begin{aligned}
4[2x-(x-2)] &= -3(3-2x) \\
4[2x-x+2] &= -9+6x \\
4[x+2] &= -9+6x \\
4x+8 &= -9+6x \\
8+9 &= 6x-4x \\
17 &= 2x \implies x = 17/2.
\end{aligned}$$

Remark. If some of the coefficients in a linear equation are rational numbers having some denominators, it is often convenient to first multiply both sides of the equation by the least common denominator (L.C.D) of all the denominators; this will result in a new equation which has integer coefficients and which has the same solution as the original equation does.

(6) Solve for x: $\quad \frac{3}{2}x - \frac{4}{3} = 20 + \frac{1}{6}x.$

$$\begin{aligned}
\frac{3}{2}x - \frac{4}{3} &= 20 + \frac{1}{6}x \\
6 \times \left(\frac{3}{2}x - \frac{4}{3}\right) &= 6 \times \left(20 + \frac{1}{6}x\right) \\
9x - 8 &= 120 + x \\
8x &= 128 \implies x = 128/8 = 16.
\end{aligned}$$

Applications: Word Problems

In mathematics, the term **word problem** is often used to refer to any math exercise where significant background information on the problem is presented as *text* rather than in *mathematical notation*. For instance a problem in mathematical notation like[1]:

solve for J:

$J = A - 20$

$J + 5 = (A+5)/2$

might be presented in a word problem as follows:

[1]This is in fact an example of a “system” of linear equations, which will be studied later in this course.

John is twenty years younger than Amy, and in five years' time he will be half of her age. What is John's age now?

The answer to the word problem is that John is 15 years old, while the answer to the mathematical problem is $J = 15$ (and $A = 35$).

To solve these problems, we look for statements in the problems that describe quantities that are equal. Then, we use algebra to write an equation that can be solved. It is customary to use variables that make it easier to remember what you're looking for, however, you can still use x as the unknown.

Examples

(7) The sum of twice a number and 13 is 75. Find the number.

Solution: The word **is** means equals, and the word **and** means plus. Therefore, we can rewrite the problem as follows:

The sum of twice a number plus 13 equals 75. Find the number.

Using numbers and a variable that represents something, N in this case (for *number*), we can write an equation that means the same thing as the original problem:

$$2x + 13 = 75.$$

Now we solve this equation by isolating the variable:

$$2x + 13 = 75 \implies 2x = 75 - 13 = 62 \implies x = 62/2 = 31.$$

(8) Find a number which, decreased by 18, is 5 times its opposite.

Solution: Again, you look for words that describe equal quantities. **Is** means equals, and **decreased by** means minus. Also, **opposite** always means negative. Keeping that information in mind, we can write the following equation

$$N - 18 = 5(-N),$$

which describes the original problem, and it is indeed really easy to solve:

$$N - 18 = -5N \implies 6N = 18 \implies N = 3.$$

Exercises

1. Solve the equations.

   (a) $4(x-1) = x+17$

   (b) $7m - 32 = 10 - 2(3+m)$

   (c) $16 + 0.55x = 0.75(x+20)$

   (d) $5 - (8+2X) = 7 + 2(X-3)$

   (e) $0.22(x+6) = 0.2x + 1.8$

   (f) $2a - 5 = 4(3a+1) - 2$

   (g) $3(2y+1) - 2(y-2) = 3$

   (h) $2[2x - (x+3)] = -2(x+3)$

   (i) $5 + 3[1 + 2(2x-3)] = 6(x+5)$

   (j) $-2[4 - (3b+2)] = 5 - 2(3b+6)$

   (k) $0.3(t+15) + 0.4(t+25) = 21.5$

   (l) $9(v+1) - 3v = 2(3v+1) - 8$

   (m) $\sqrt{2}x - 2 = 0$

   (n) $\sqrt{3}x - 1 = 5 - 3\sqrt{3}$

   (o) $3\sqrt{2} - \sqrt{6}x = \sqrt{8}$

   (p) $\frac{3}{4} - \frac{3}{5}x = \frac{19}{20}$

   (q) $\frac{3}{8} - \frac{1}{2}(x-1) = 2x$

   (r) $\frac{1}{2}(x+2) + \frac{3}{4}(x+4) = x+5$

   (s) $\frac{1}{3}(x-1) - \frac{3}{5} = \frac{11}{15} - x$

   (t) $\frac{3}{4} - \frac{1}{8}(1-3x) = 2x+1$

   (u) $\frac{x-2}{3} = \frac{x+3}{-2}$

   (v) $\frac{5+t}{10} - t = \frac{1}{2}$

   (w) $\frac{x-2}{3} + \frac{3x}{5} = 4$

   (x) $\frac{0.6x - 1.29}{0.33} - 3.67 = -6.67$

2. If $4 - 3a = 7 - 2(2a+5)$, evaluate $3a^2 - 2$.

3. If $3[2 - 4(x-1)] = 3(2x+8)$, find $-6x^2 + 2x$.

4. The difference between four times a number and 13 is 55. Find the number.

5. The difference between 131 and twice a number is 45. Find the number.

6. Three times the difference of a number and 12 is 12. Find it.

7. The sum of twice a number and 11 is 53. Find it.

8. The sum of two numbers is 21. Three times the larger is equal to four times the smaller. Find them.

9. The sum of three consecutive odd integers is fifty one. Find them.

10. Find three consecutive odd integers such that three times the middle one is one more than the sum of the first and the third.

11. Find three consecutive even integers whose sum is -18.

12. Twice the smallest of three consecutive even integers is 6 more than the largest one. Find the integers.

13. Find the number such that 5 times three more than itself is 80.

14. Four times the sum of twice a number and 23 is 76. Find the number.

15. A wallpaper hanger charges a fee of $25 plus $12 per roll of wallpaper. How many rolls were used if the total charge for hanging wallpaper is $97?

16. A mobile phone company charges $13.99 monthly for the first 200 messages and $0.12 for each text messaging over 200 in 1 month. Find the number of text messages of Sara in April whose bill was $16.63 before taxes.

17. A taxi charges $3.50 initially and $2.80 per kilometer. How many kilometers did a customer travel if his total taxi fare was $17.50.

18. A grant of $12,000 is to be divided into 3 scholarships. How much is given to each scholarship if the second is double the first and the third is $1700 more than the second?

19. A man had $185 in five and ten dollar bills. How many of each did he have if he had 7 fives more than tens?

20. A child has $11.35 in his piggy bank in nickels, dimes and quarters. How many of each does he have if the number of his dimes is half of the number of his nickels and the number of quarters is 5 less than the number of his dimes?

2.2 Formulae

Roughly speaking, any mathematical relationship involving two or more variables is called a formula.

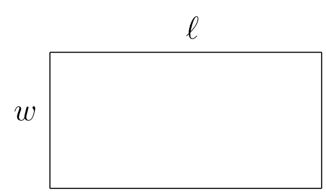
Examples

(1) The formula expressing the area A of a rectangle in terms of its length ℓ and its width w is

$$\boxed{A = \ell \cdot w}$$

And the formula expressing the perimeter P in terms of ℓ and w is

$$\boxed{P = 2(\ell + w)}$$



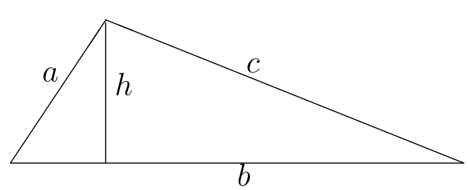
(2) In case of a triangle with the three sides $a,\ b$ and c, the formulas expressing the area A and the perimeter P in terms of the sides are respectively

$$\boxed{A = \frac{1}{2}bh = \frac{bh}{2}}$$

and

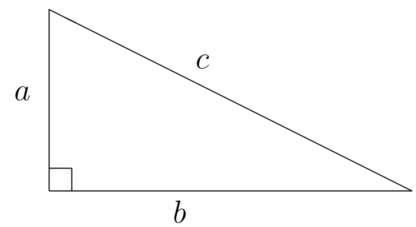
$$\boxed{P = a + b + c}$$

where h is the hight corresponding to the base b.

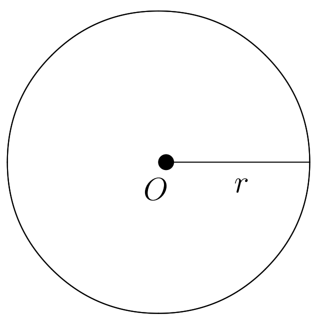


(3) In case of a right-angled triangle with the legs a and b and **hypotenuse** c, the **Pythagorean Formula** is given by

$$\boxed{a^2 + b^2 = c^2}$$



(4) The area A and the circumference C of a circle of radius r are obtained by the following formulas:



$$\boxed{C = 2\pi r}$$

$$\boxed{A = \pi r^2}$$

Assigning Values to the Variables

If the values of all but one variable in a formula are known, one can find the value of the unknown variable simply by substituting the given values for the known variables. Here is one example.

(5) In the formula $A = \frac{1}{2}bh$, if $A = 450$ and $h = 9$, find the value of b.

Solution. We have $450 = \frac{1}{2}b \cdot 9$ from which we find $b = 100$.

Solving for one Particular Variable

Another typical problem that could be asked is to "solve" for one particular variable. The idea is to use basic algebraic operations (adding or subtracting, multiplying, dividing, moving terms, etc) to "isolate" the required variable. We shall give two examples.

Examples

(6) In the formula $A = \frac{1}{2}h(b+B)$ solve for h.

Solution. We have

$$A = \frac{1}{2}h(b+B) \implies 2A = h(b+B) \implies h = \frac{2A}{b+B}.$$

(7) Solve $A = \frac{1}{2}h(b+B)$ for b.

Solution. We have

$$A = \frac{1}{2}h(b+B) \implies 2A = h(b+B) \implies b+B = \frac{2A}{h} \implies b = \frac{2A}{h} - B.$$

Exercises

1. Find the value of the unknown variable in the following formulas.

   (a) $A = \frac{1}{2}bh$ if $b = 11$ and $h = 5.61$

   (b) $P = 2l + 2w$ if $l = 3.75$ and $P = 12.26$

   (c) $C = 2\pi r$ if $C = 176.12$

   (d) $F = \frac{9}{5}C + 32$ if $F = 76$

   (e) $K = \frac{1}{2}h(a + b)$ if $K = 48$, $a = 5$ and $b = 7$

   (f) $C = a - 2\frac{A}{b}$ if $a = -2$, $A = 3$ and $b = 7$

   (g) $a^2 = b^2 + c^2$ if $b = 3$ and $c = 4$

   (h) $a = \sqrt{b^2 + c^2}$ if $c = -15$ and $b = -20$

   (i) $V = \pi r^2 h$ if $V = 14.75$ and $r = 2.1$

2. Solve each formula for the indicated variable.

   (a) $P = a + b + c$ for b

   (b) $P = 2l + 2w$ for w

   (c) $V = lwh$ for h

   (d) $y = \frac{1}{2}(x + z)$ for x

   (e) $A = P(1 + r)$ for r

   (f) $y = mx + b$ for m

   (g) $G = 2b(R - r)$ for r

   (h) $A = P + Prt$ for t

   (i) $A = \frac{1}{2}h(b + B)$ for h

(j) $A = \frac{1}{2}h(b+B)$ for b

(k) $A = \frac{1}{2}hb + B$ for b

(l) $F = \frac{9}{5}C + 32$ for C

(m) $C = 1 - \frac{A}{b}$ for A

(n) $\frac{x}{2} + y = z^2$ for x

(o) $F = \frac{GmM}{d^2}$ for m

(p) $a^2 = b^2 + c^2$ for b

3. Find the radius of a circle whose circumference is 14.32 cm.

4. Find the dimensions of a rectangle of perimeter 26 cm whose length is 2 cm bigger than its width.

5. What is the height of a triangle whose area is 48 ft^2 and its base is double of its height?

6. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ where h and r stand for the height and the radius of the cone respectively. Find the height of a cone whose volume is 36.65 cubic inches and its radius is 4.2 in.

2.3 Final Answers to Exercises

Section 2.1

1. (a) $x = 7$
   (b) $m = 4$
   (c) $x = 5$
   (d) $x = -1$
   (e) $x = 24$
   (f) $a = \dfrac{-7}{10}$
   (g) $y = -1$
   (h) $x = 0$
   (i) $x = \dfrac{20}{3}$
   (j) $b = \dfrac{-1}{4}$
   (k) $t = 10$
   (l) No Solution.
   (m) $x = \sqrt{2}$
   (n) $x = 2\sqrt{3} - 3$
   (o) $x = \dfrac{\sqrt{3}}{3}$
   (p) $x = \dfrac{-1}{3}$
   (q) $x = \dfrac{7}{20}$
   (r) $x = 4$
   (s) $x = \dfrac{5}{4}$
   (t) $x = \dfrac{-3}{13}$
   (u) $x = -1$
   (v) $t = 0$
   (w) $x = 5$
   (x) $x = \dfrac{1}{2}$
2. 145
3. $\dfrac{-4}{3}$
4. 17
5. 43
6. 16
7. 21
8. 12
9. 15, 17 and 19
10. -1, 1 and 3

11. $-8, \ -6$ and -4

12. $8, \ 10$ and 12

13. 13

14. -2

15. 6

16. 22

17. 5

18. 2060, 4120 and 5820 respectively.

19. 10 (\$10 bill) *and* 17 (\$5 bill.)

20. 56 nickels, 28 dimes and 23 quarters.

Section 2.2

1. (a) $A = 30.855$
   (b) $w = 2.38$
   (c) $r = 28.03$
   (d) $C = 24.44$
   (e) $h = 8$
   (f) $C = \dfrac{-20}{7} = -2.86$
   (g) $a = \pm 5$
   (h) $a = 25$
   (i) $h = 1.065$

2. (a) $b = P - a - c$
   (b) $w = \dfrac{P - 2l}{2}$
   (c) $h = \dfrac{V}{lw}$
   (d) $x = 2y - z$
   (e) $r = \dfrac{A}{p} - 1 = \dfrac{A - p}{p}$
   (f) $m = \dfrac{y - b}{x}$
   (g) $r = R - \dfrac{G}{2b}$
   (h) $t = \dfrac{A - P}{Pr}$
   (i) $h = \dfrac{2A}{b + B}$
   (j) $b = \dfrac{2A}{h} - B$
   (k) $b = \dfrac{2(A - B)}{h}$
   (l) $C = \dfrac{5}{9}(F - 32)$
   (m) $A = b(1 - C)$
   (n) $x = 2(z^2 - y)$

(o) $m = \dfrac{Fd^2}{GM}$

(p) $b = \pm\sqrt{a^2 - c^2}$

3. $r = 2.28$

4. $w = 5.5,\ l = 7.5$

5. $h = 6.93$

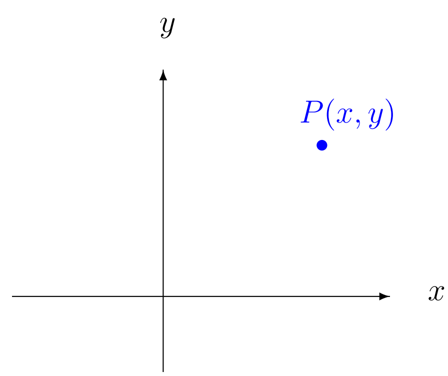
6. $h = 1.98$

Chapter 3

Cartesian Plane and Graphing

3.1 Cartesian Plane

The **Cartesian plane** or the **rectangular coordinate system** is formed by considering the right-angle intersection of a horizontal real number line called the x-axis and a vertical real number line called the y-axis. The coordinate system is often also referred to as the *Cartesian plane.*



This yields a one-to-one correspondence between the points P in the plane (geometry) and the ordered pairs of real numbers (x, y) (algebra), where x and y are the *coordinates* of the point P. In other words, we get a "dictionary" between geometry and algebra.
For instance, the phrase "consider the point $(1, 2)$" is understood in this context as "consider the specific point in the plane which corresponds to the pair of real numbers $(1, 2)$. Or conversely, when we say "the circle $x^2 + y^2 = 25$ passes through the point $(3, 4)$" what we really mean is that there is a circle the coordinates of all of whose points satisfy the given

equation, and it is this circle which passes through the point in the plane whose coordinates are $(3, 4)$.
The x-axis and and y-axis together divide the Cartesian plane into four partes called **Quadrants**. Both coordinates of a point which lies in Quadrant I are positive. In Quadrant II, only the y-coordinate is positive, etc. The x of a point is zero if and only if it lies on the y-axis; similarly, the y of a point is zero if and only if it lies on the x-axis. The unique point whose both coordinates are zero is denoted by $O(0, 0)$ and will be called the **origin**.

Distance Formula Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the length of the line segment joining A and B is given by the formula

$$\boxed{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

Examples

(1) The distance between the points $A(-4, 7)$ and $B(1, -5)$ is

$$d = \sqrt{((-4) - 1)^2 + (7 - (-5))^2} = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13.$$

(2) The distance between $C(2, 3)$ and $D(1, 1)$ is equal to

$$d = \sqrt{(2 - 1)^2 + (3 - 1)^2} = \sqrt{1 + 4} = \sqrt{5}.$$

Midpoint Formula The middle point M of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, which is equidistant from both ends, is called the **midpoint**. The coordinates of M are given by the following formulas:

$$\boxed{x_M = \frac{x_1 + x_2}{2}, \quad \text{and} \quad y_M = \frac{y_1 + y_2}{2}.}$$

Examples

(3) The midpoint of the line segment joining $A(4, 1)$ and $B(6, -3)$ is

$$M(x_M, y_M) = \left(\frac{4 + 6}{2}, \frac{1 + (-3)}{2}\right) = (5, -1).$$

(4) Find the other endpoint of the line segment with one endpoint as $A(-1,-1)$ and with the midpoint as $M(1/2, 1)$.
Solution. Calling the other endpoint $B(x_B, y_B)$, we have

$$x_M = 1/2 = \frac{-1+x_B}{2} \Longrightarrow -1+x_B = 1 \Longrightarrow x_B = 2.$$

And in a similar manner, one finds $y_B = 3$.

(5) Find the point(s) on the y-axis that are a distance of 5 from the point $(3, 5)$.

Solution. A typical point on the y-axis can be expressed as $(0, y)$ (why?). So our problem is to find y such that

$$\sqrt{(0-3)^2 + (y-5)^2} = 5.$$

This is a rather simple radical equation whose solutions are $y_1 = 1$ and $y_2 = 9$ and thus the points are $(0, 1)$ and $(0, 9)$.

Exercises

1. Find the distance between the given points.

   (a) $(2,1)$ and $(6,7)$

   (b) $(-1,1)$ and $(2,3)$

   (c) $(3,7)$ and $(-7,-3)$

   (d) $(-5,0)$ and $(-1,-3)$

   (e) $(\frac{1}{2},\frac{-5}{3})$ and $(\frac{-3}{2},\frac{-2}{3})$

   (f) $(\frac{2}{3},-3)$ and $(\frac{-1}{2},-5)$

   (g) $(\sqrt{2},-2)$ and the origin

   (h) $(3\sqrt{3},-2)$ and $(\sqrt{3},-2)$

   (i) $(a,-b)$ and $(-a,b)$

2. Find the midpoint of the line segment between the points.

   (a) $(4,3)$ and $(-2,5)$

   (b) $(-6,-2)$ and $(-3,5)$

   (c) $(-4,7)$ and the origin.

   (d) $(-5,0)$ and $(-2,\frac{5}{2})$

   (e) $(\frac{3}{2},-3)$ and $(\frac{5}{3},\frac{1}{2})$

   (f) $(2\sqrt{3},-2\sqrt{5})$ and $(\sqrt{27},\sqrt{20})$

   (g) $(\sqrt{2},-5)$ and $(3\sqrt{2},3)$

   (h) $(a+b,b-a)$ and $(a-b,b+a)$

3. Find the end point of the line segment whose midpoint is $(-1,3)$ and the other end point is $(-5,4)$.

4. If the end point of a line segment is given by $(\frac{2}{3}, \frac{5}{2})$ and its midpoint by $(-3, \frac{1}{2})$, find the other end point.

5. Find the perimeter of the triangle whose vertices are $(-2, 1)$, $(4, 3)$ and $(1, -3)$.

6. Find the perimeter of the rectangle whose vertices are $(-2, 5)$, $(3, 5)$, $(3, -4)$ and $(-2, -4)$.

7. Find the area of the rectangle whose vertices are $(\frac{2}{5}, 2)$, $(\frac{2}{5}, -1)$, $(0, 2)$ and $(0, -1)$.

8. Find the area of the circle whose center is located at $(-1, 3)$ and passes through the point $(2, -3)$.

9. Show that the point $(-3, 2)$ is equidistance from $(-4, -2)$ and $(1, 1)$.

10. Find the point on the $y-$axis whose distance to the point $(3, 5)$ is 5 unit.

11. Find the point on the $x-$axis that is equidistance from the points $(1, -2)$ and $(3, 1)$.

12. Find y such that the point $(4, y)$ is equidistance from $(-1, -1)$ and $(1, 3)$.

13. Find x if the distance between $(x, -1)$ and $(2, 1)$ is $2\sqrt{2}$.

14. Find y if the distance between $(-3, y)$ and $(2, 8)$ is $5\sqrt{2}$.

15. Show that the points $A(1, -3)$, $B(8, -7)$ and $C(5, -1)$ form a right-angled triangle.

3.2 Introduction to Functions

Definition 3.1 *A function is a rule or an assignment f which to any member x of a set called* **domain** *corresponds* **one and only one** *member y of another set called* **range**. *Therefor each* **input** *x uniquely determines one* **output** *y.*
This assignment is traditionally denoted by $y = f(x)$. x is known as the **independent** *variable, whereas y is called the* **dependent** *variable.*

Examples

(1) The function f which to any person it assigns its age:

$$f(\text{a person}) = \text{his/her age.}$$

The domain of f is the set of all people, and its range is the set of all possible ages.

(2) The function g which assigns to any triangle Δ its area:

$$g(\Delta) = \text{the area of } \Delta.$$

The domain of g is the set of all triangles and its range is the set of all positive numbers (why?)

(3) For each person, his height is a function of his age, however, his age is not a function of his height. (explain why?)

Representing Functions. There are six different ways to introduce functions:

(I) By Words
A function can simply be described by words. For example, the function u which to any number assigns 5 more than twice that number. For instance, $u(10) = 5 + 2 \times 10 = 25$.
Or, the function which to any country corresponds its currency. For instance, g(Canada)= Canadian dollar.

(II) By Arrows
We may use an arrow to indicate the assignment $y = f(x)$:

$$\boxed{x \longrightarrow f(x) \quad \text{or} \quad x \xrightarrow{f} y}$$

For instance, with f as the "age function", if John is 33 years old, we may write

$$\text{John} \xrightarrow{f} 33.$$

(III) By Ordered Pairs

We may consider a function as a collection of ordered pairs (x, y), where y is what f corresponds to x, that is to say, an ordered pair (x, y) belongs to f if $y = f(x)$. For example if we write

$$h = \{..., (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), ...\}$$

we understand that h is the function with the set of integers as its domain which sends any integer number to its square:

$$..., h(-3) = 9, h(-2) = 4, h(-1) = 1, h(0) = 0, h(1) = 1, h(2) = 4, h(3) = 9, ...$$

We should remark that a collection of ordered pairs need not necessarily represent a function; for example the collection $R = \{(1, 2), (2, 3), (3, 4), (1, 5)\}$ is not a function since $x = 1$ corresponds to two **distinct** y's; in other words, an ambiguity will arise if one writes $R(1)$ (why?)

(IV) By Tables

The following table represents the input, number of the month (January=1,...,December=12) and the output is the total precipitation in mm for each month in the year 2013, in Ottawa.

input: x	1	2	3	4	5	6	7	8	9	10	11	12
output: $y = f(x)$	40.4	52.4	43.4	101.5	87.0	131.0	109.0	113.8	81.8	80.6	84.7	56.4

For instance the precipitation of February was 52.4 mm or in other words $f(2) = 52.4$.

The following table defines the function $g(n) = y$ where n is the age of the kids in primary school and y represents their average height in inches.

input: n	5	6	7	8	9	10	11
output: $y = g(n)$	41.0	42.5	43.4	45.2	48.0	51.7	54.1

Based on this table the average height of the 10 year old students in grade 5 is 51.7 inches and we can write it as $g(10) = 51.7$

(V) By Formula
When x and y are both numbers, most often the assignment which defines a function can be described by a formula relating x and y. For example, the function u introduced earlier can be written as $y = u(x) = 5 + 2x$; or the formula of the function h is $y = h(x) = x^2$. *Expressing functions by formulas has the widest applications in mathematics!*

(VI) By Graph
Another useful way of giving functions is by their graphs. *The graph of a function f is by definition the collection of all points (x, y) in the Cartesian plane that correspond to the ordered pairs $(x, f(x))$ of the function where $y = f(x)$.*
We should remark here as well that the graph of a function can not contain two points on a vertical line. This observation is known as the **Vertical Line Test**.

Examples

(4) Given the function $f(x) = 3x^2 + 2x - 4$, find
(a) $f(-1)$; **(b)** the value of x when $f(x) = 12$; **(c)** $f(a)$; **(d)** $f(x + 1)$; **(e)** $f(x + t)$.

Solution.

(a) We plainly have $f(-1) = 3(-1)^2 + 2(-1) - 4 = -3$;
(b) We need to solve the equation $3x^2+2x-4 = 12$. The solutions are $x_1 = 2$ and $x_2 = -8/3$;
(c) We obviously have $f(a) = 3a^2 + 2a - 4$;
(d) Similarly, we note $f(x + 1) = 3(x + 1)^2 + 2(x + 1) - 4 = 3x^2 + 8x + 1$;
(e) And finally, we see $f(x + t) = 3(x + t)^2 + 2(x + t) - 4 = 3x^2 + 6tx + 2x + 3t^2 + 2t - 4$.

(5) Find the domain of $f(x) = \dfrac{15x - 7}{x^2 - 3x}$.

Solution. For a given value of x, the only condition for $f(x)$ to make sense is $x^2 - 3x \neq 0$. But $x^2 - 3x = 0$ only when $x = 0,\ 3$. Thus, the domain of f is all real numbers with 0 and 3 removed, namely

$$D_f = \mathbb{R} \setminus \{0, 3\}$$

or

$$D_f = (-\infty, 0) \cup (0, 3) \cup (3, +\infty).$$

(6) Find the domain of $f(x) = \sqrt{4-2x}$.

Solution. Here the condition is $4-2x \geq 0$ and this happens if $x \leq 2$. Therefore, we conclude that

$$D_f = (-\infty, 2].$$

(7) Find and simplify $\dfrac{f(x+h)-f(x)}{h}$ if $f(x) = x^2 - 5x + 3$.

Solution. We have

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\big((x+h)^2 - 5(x+h) + 3\big) - (x^2 - 5x + 3)}{h} \\
&= \frac{(x^2 + 2xh + h^2) - \cancel{5x} - 5h + \cancel{3} - x^2 + \cancel{5x} - \cancel{3}}{h} \\
&= \frac{\cancel{x^2} + 2xh + h^2 - 5h - \cancel{x^2}}{h} \\
&= \frac{h(2x+h-5)}{h} \\
&= 2x + h - 5.
\end{aligned}$$

Exercises

1. Determine whether each collections of ordered pairs is a function:

    (a) $\{(-1,3),(2,2),(3,4),(0,0)\}$

    (b) $\{(0,-3),(-2,2),(3,-4),(-3,0)\}$

    (c) $\{(-7,5),(3,5),(0,-4),(-1,6),(7,5)\}$

    (d) $\{(4,3),(-2,2),(-3,4),(-2,0),(1,-1),(5,-1)\}$

    (e) $\{(1,3),(2,3),(\sqrt{3},3),(-5,3),(-11,3)\}$

2. Determine which table represent y as a function of x.

    (a)

x	1	3	7	11	22	25	27
y	3	8	1	0	8	7	3

    (b)

x	1	3	3	4	5	7	11
y	-3	8	14	0	2	7	1

    (c)

y	-1	3	3	4	5	7	11
x	3	8	14	0	2	7	1

    (d)

x	y
1	-3
4	-6
9	-9
16	-12

    (e)

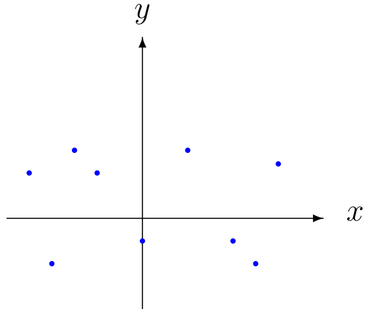
x	y
1	-3
2	-3
5	-3
7	-3

    (f)

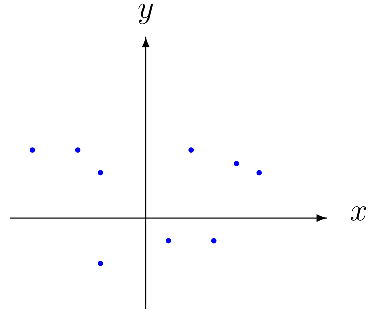
y	x
7	-3
12	-3
81	-9
16	-10

3. Determine which graph represents a function.

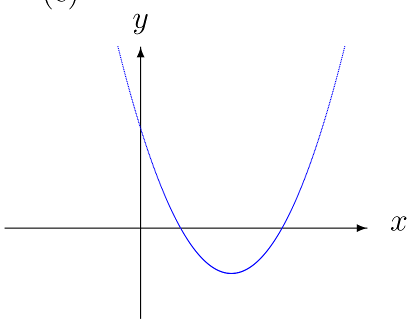
    (a)



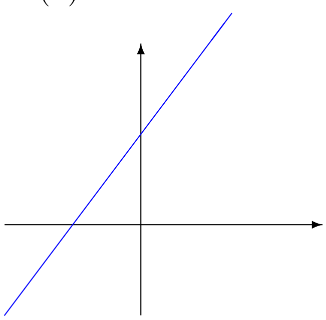
    (b)



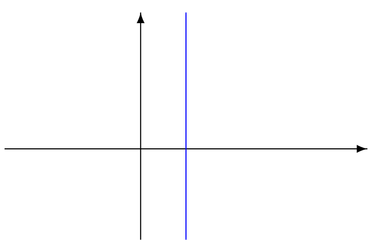
(c)



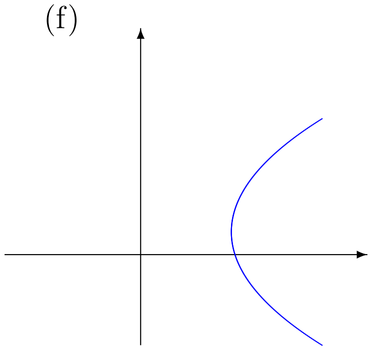
(d)



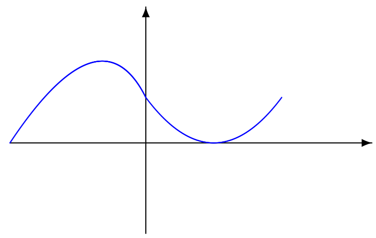
(e)



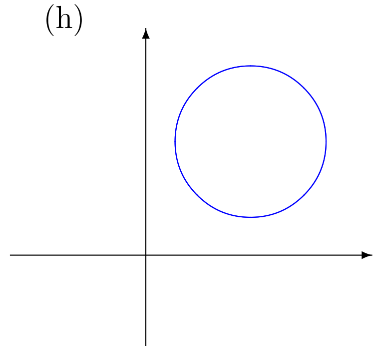
(f)



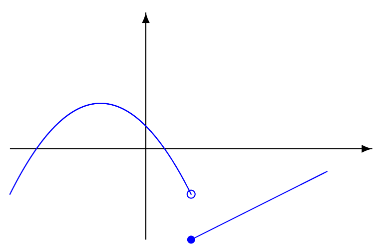
(g)



(h)



(i)



4. Find the domain and the range of the following functions.

   (a) $\{(-1,3),(2,2),(3,4),(0,0),(-3,3),(-7,3)\}$

   (b) $\{(0,-3),(-2,-3),(3,-3),(-3,-3),(5,-3)\}$

(c) $\{\cdots, (-3,3), (-2,2), (-1,1), (0,0), (1,-1), (2,-2), (3,-3), \cdots\}$

5. Let $f = \{(-4,2), (2,2), (3,-4), (0,-3), (-3,1), (-2,-2), (1,0)\}$. find **(a)** $f(0)$, **(b)** $f(-4)$, **(c)** $f(f(-3))$, **(d)** Find x where $f(x) = 1$.

6. The function $y = g(x)$ is given by the following table:

x	2	3	5	7	11	13	17	19	23
$y = g(x)$	1	2	4	8	16	32	64	128	256

Find **(a)** $g(3)$, **(b)** $g(13)$, **(c)** $g(g(3))$, **(d)** Find x where $g(x) = 128$.

7. The function $y = h(n)$ is given by the following table:

n	0	1	3	7	9	11	14	17	20
$y = h(n)$	1	2	2	8	6	2	8	7	6

Find **(a)** $h(3)$, **(b)** $h(11)$, **(c)** $h(h(17))$, **(d)** Find n where $h(n) = 6$.

8. Find the domain and the range of the functions g and h given with the tables above.

9. For each of the following functions evaluate $f(-2)$, $f(-1)$, $f(0)$ and $f(3)$ where possible:

(a) $f(x) = 5 - 2x$

(b) $f(x) = 3x^2 - 2x + 8$

(c) $f(x) = -x^3 + 2x$

(d) $f(x) = (x-2)(x+3)$

(e) $f(x) = \dfrac{x-2}{x-3}$

(f) $f(x) = 2^x$

(g) $f(x) = \sqrt{x+2} - 3$

(h) $f(x) = \sqrt{x-2}$

(i) $f(x) = \dfrac{2x-1}{x^2+3}$

(j) $f(x) = 3^{x-1}$

10. Complete each ordered pairs (x, y) for each function.

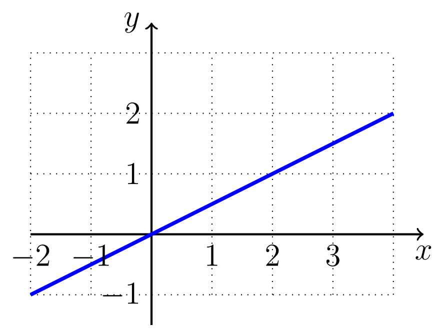
(a) $y = \dfrac{1}{x}$ $(-1, \ )$, $(\ , 3)$

(b) $y = -3x + 5$ $(-2, \ )$, $(\ , 11)$

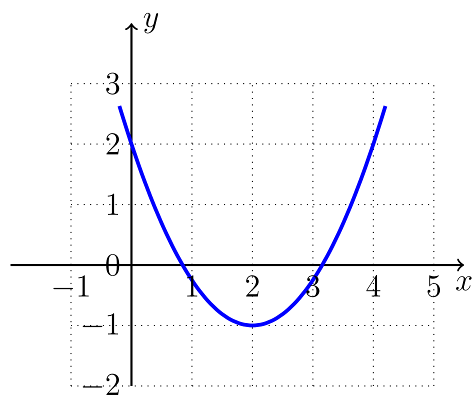
(c) $y = x^2 - x$ $(2, \ )$, $(\ , 0)$

(d) $y = \sqrt{x-3}$ $\qquad$ $(3, \ )$, $\ (\ , 3)$

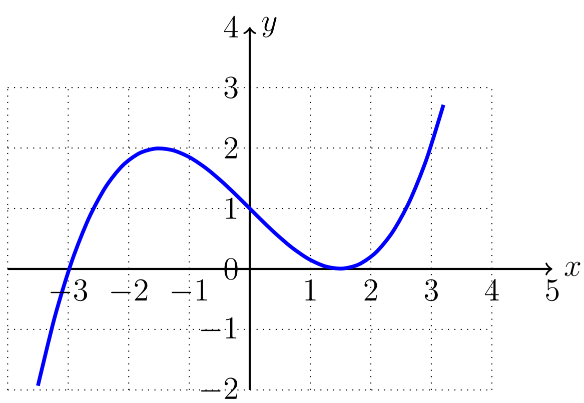
11. Given the graph of the linear function f, evaluate **(a)** $f(0)$, **(b)** $f(2)$ and **(c)** solve $f(x) = 0$ for x.



12. Given the graph of f below, evaluate **(a)** $f(2)$, **(b)** $f(0)$ and **(c)** solve $f(x) = 2$ for x.



13. Given the graph of the function g below, evaluate **(a)** $g(3)$, **(b)** $g(-1)$ and **(c)** solve $g(x) = 0$ for x.



14. If $f(x) = 7x - 2$, evaluate **(a)** $f(-3)$, **(b)** $f(11)$, **(c)** $f(f(0))$ and **(d)** solve $f(x) = 4$ for x.

15. If $g(x) = 3x^2 + 5x - 2$, evaluate **(a)** $g(0)$, **(b)** $g(-1)$, **(c)** $g(g(1))$ and **(d)** solve $g(x) = 0$ for x.

16. If $f(x) = x + 1$ and $g(x) = x^2$, evaluate **(a)** $g(3)$, **(b)** $f(-1)$, **(c)** $f(-2) + g(3)$, **(d)** $(f(3))^2$, **(e)** $2g(-2) + 3f(-1)$, **(f)** $g(-1)g(3)$, **(g)** $\dfrac{f(3)}{g(-2)}$.

17. Let $f(x) = -3x + 5$, evaluate **(a)** $f(-3)$, **(b)** $f(a)$, **(c)** $f(a + 2)$ and **(d)** $f(a) + f(2)$.

18. Let $g(x) = \dfrac{x-1}{2}$, evaluate **(a)** $g(g(1))$, **(b)** $g(x+1)$, **(c)** $g(x-1)$ and **(d)** $g(x) - g(1)$.

19. Let $f(x) = \dfrac{x-2}{x+1}$, evaluate **(a)** $f(x+2)$, **(b)** $f(\frac{1}{x})$, **(c)** $f(x) + 1$ and **(d)** $f(x) + f(1)$.

20. Find the domain of the following functions.

    (a) $f(x) = -3x + 2$

    (b) $f(x) = 2x^2 - 3x + 1$

    (c) $g(x) = x^2 - x + 7$

    (d) $h(x) = \dfrac{2x-1}{5}$

    (e) $f(x) = \dfrac{2x}{x-3}$

    (f) $g(x) = \dfrac{x^2 - 3x}{2x+1}$

    (g) $f(x) = \sqrt{x-1}$

    (h) $h(x) = \sqrt{2-x}$

    (i) $f(x) = \dfrac{\sqrt{x}}{x-1}$

    (j) $f(x) = \dfrac{2x}{\sqrt{x-1}}$

    (k) $g(x) = \dfrac{x-1}{\sqrt{x+2}}$

3.3 Linear Functions; Slope and Equation of a Line

Definition 3.2 *A function that can be expressed in the following form*

$$f(x) = ax + b, \qquad (a, b \in \mathbf{R}),$$

is called a **linear function**.

The main reason for being named so is that the graph of such function is a (straight) non-vertical line (explain why non-vertical?)
Such line will always cross the y axis, and the point of intersection is called the y**-intercept.** It, however, might or might not have an x-intercept depending on whether its graph is horizontal or not. In case the line is not horizontal, the point of intersection with the x-axis is accordingly called the x**-intercept**. One of the significances of the intercepts is that one can easily sketch the graph of a linear function as all one needs is no more than two points!
The domain of a linear function $f(x) = ax + b$ where $a \neq 0$ as well as its range is the set of all real numbers: $\mathbf{R} = (-\infty, +\infty)$. In the case $a = 0$, the domain is the set of all real numbers whereas its range is only $\{b\}$.
As we know a vertical line can not be considered as a linear function (why?), however, one can still study vertical lines as a category of lines. The **general equation** for lines that also includes vertical lines is

$$\boxed{Ax + By = C}$$

where $A,\ B$ and C are real numbers and at least one of A or B is not zero.

Examples

(1) For a linear function f, if $f(0) = 3$ and $f(2) = 13$, find

(a) the formula of f; **(b)** $f(-2)$; **(c)** the value of x such that $f(x) = 15$; **(d)** sketch its graph.

Solution.

(a) Setting $f(x) = ax + b$, we have

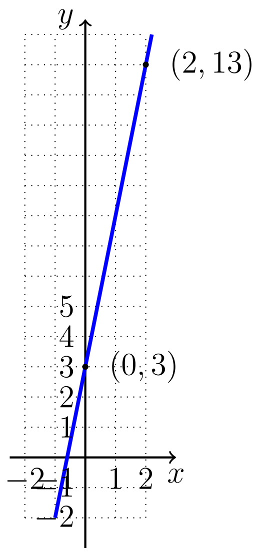
$$a \times 0 + b = 3 \qquad \text{and} \qquad a \times 2 + b = 13,$$

from which we get $a = 5$ and $b = 3$. Thus $f(x) = 5x + 3$ is the formula of f;

(b) Now obviously we have $f(-2) = 5(-2) + 3 = -7$;

(c) By solving $5x + 3 = 15$ for x one easily finds $x = 12/5$;

(d) Plot the points $(0, 3)$ and $(2, 13)$, and then join them by a straight line!



The Slope and Equations of a Line

Definition 3.3 *The* **slope** *of a line describes its steepness or incline. A higher slope value indicates a steeper incline. The slope is defined as the ratio of the "rise" divided by the "run" between any two points on a line, or in other words, the ratio of the altitude change to the horizontal distance between any two points on the line. Thus, given two points* (x_1, y_1) *and* (x_2, y_2) *on a line, the slope* m *of the line is*

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

Slope-Point Formula

The equation

$$\boxed{y - y_1 = m(x - x_1)}$$

describes the line whose slope is m and which passes through the point (x_1, y_1). In particular, if the given point happens to be the y-intercept $(0, b)$ of the line, the equation above takes

the following **standard** form:

$$\boxed{y = mx + b}$$

Examples

(2) The slope of the line through the points $(2, -3)$ and $(-1, 6)$ is

$$m = \frac{(-3) - 6}{2 - (-1)} = -3.$$

(3) An equation of the line through the point $(1, 5)$ with slope 3 is

$$y - 5 = 3(x - 1) \qquad \text{or} \qquad y = 3x + 2.$$

(4) An equation of the line with slope 4 whose y-intercept is $(0, -2)$ is

$$y = 4x - 2.$$

Parallel/Perpendicular Lines

If m_1 and m_2 are the slopes of two lines, the necessary and sufficient condition for the two lines to be

- parallel is that $\boxed{m_1 = m_2}$;
- perpendicular is that $m_1 m_2 = -1$, or equivalently that $\boxed{m_2 = -\frac{1}{m_1}}$.

Examples

(5) Use slopes to show that the three points $A(3, 8)$, $B(1, 5)$ and $C(-1, 2)$ are co-linear, that is to say, they all lie on one line.

Solution. It will suffice to show that the slope of the line through A and B is equal to the slope of the line through B and C (explain why?):

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{5 - 8}{1 - 3} = 3/2, \qquad m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{2 - 5}{-1 - 1} = 3/2,$$

(6) The two lines $2x - 4y + 6 = 0$ and $2x + 9 = -y$ are perpendicular since their slopes are reciprocal-opposite of each other:

$$m_1 = 1/2, \qquad m_2 = -2.$$

(7) An equation of the line through $(-4, 3)$ and $(-2, 5)$ is

$$y - 5 = \frac{3-5}{(-4)-(-2)}(x - (-2)) \qquad \text{or} \qquad y = x + 7.$$

(8) It is known that the rate at which crickets chirp is linearly related to the air temperature. At 59° F, they chirp 76 times per minute, and at 65° F, they chirp 100 times per minute.

(a) Find the linear equation relating the chirping rate y to the air temperature x in ° F;

(b) What is the chirping rate at 90° F?

(c) What would you predict the temperature to be if you counted 120 chirps during 1 minute?

(d) Below what temperature would the crickets be silent?

Solution.

(a) $y - 100 = \dfrac{100-76}{65-59}(x - 65) = 4(x - 65) \implies y = 4x - 160$

(b) $y = 4(90) - 160 = 200$

(c) $120 = 4x - 160 \implies x = 70°\text{F}$

(d) $0 = 4x - 160 \implies x = 40°\text{F}$

Exercises

1. Find the slope of the line through the given points. Write an equation of the line and graph it.

   (a) $(3, 5)$ and $(2, 7)$

   (b) $(-1, 2)$ and $(2, -4)$

   (c) $(0, 0)$ and $(-2, 4)$

   (d) $(1, -3)$ and $(2, 3)$

   (e) $(-1, -3)$ and $(1, -5)$

   (f) $(-3, 7)$ and $(0, 5)$

   (g) $(-2, -4)$ and $(-4, -2)$

   (h) $(1, -2)$ and $(-1, 1)$

   (i) $(3, 5)$ and $(-7, 5)$

   (j) $(-1, -4)$ and $(-1, 7)$

2. Find an equation of the line with slope -2 and the y-intercept $(0, 3)$.

3. Find an equation of the line that contains the point $(-1, 0)$ and has slope $\dfrac{-1}{3}$.

4. Find an equation of the line with slope $\dfrac{2}{5}$ and through the point $(2, -2)$.

5. Find an equation of the line that contains the points $(-1, 4)$ and $(2, 9)$.

6. Find an equation of the line passing through the points $(0, 0)$ and $(-3, 7)$.

7. Find an equation of the line with intercepts $(0, -2)$ and $(4, 0)$.

8. Which pair of the following lines are parallel?

   (a) $y = -3x + 2$ and $2y + 3x = 1$

   (b) $2x - y = -5$ and $y - 2x + 3 = 0$

   (c) $2x - 4y = 7$ and $y = \dfrac{1}{2}x$

   (d) $2x - 3y = 5$ and $3x - 2y = 5$

9. Determine whether the pair of lines are perpendicular or not.

   (a) $y = -3x + 2$ and $y + 3x = 1$

   (b) $2x - y = -5$ and $2y + x + 3 = 0$

   (c) $x + 3y = 7$ and $y = 3x$

   (d) $2x - 3y = 5$ and $3x + 2y = 5$

10. Find an equation of the line through the point $(-3, 4)$ and parallel to the line $y = -3x + 2$.

11. Find an equation of the line that contains the point $(3, -1)$ and is parallel to the line $2x - 3y = 5$.

12. Find an equation of the line that contains the point $(-5, -4)$ and is parallel to the x-axis.

13. Find an equation of the line through the point $(3, -4)$ and parallel to the y-axis.

14. Find an equation of the line through the point $(1, -2)$ and perpendicular to the line $y = -2x + 7$.

15. Find an equation of the line that contains the point $(-1, -4)$ and is perpendicular to the line $3x - 2y + 9 = 0$.

16. Find an equation of the line through the point $(-3, -2)$ and perpendicular to the x-axis.

17. Find an equation of the line that contains the point $(4, -1)$ and is perpendicular to the y-axis.

18. Show that the points $(3, 8)$, $(1, 5)$ and $(-1, 2)$ are co-linear.

19. Use slopes to show that the points $(-1, -3)$, $(6, 1)$ and $(2, -5)$ form a right-angled triangle.

20. Find k if the line $y = kx - 5$ passes through the point $(-2, 3)$.

21. If $(-4, 11)$, $(2, -4)$ and $(6, n)$ are coordinates of three points on the same line, determine n.

22. Find a if the lines $ax - 2y = 5$ and $y - 3x - 7 = 0$ are parallel.

23. Find k if the line $kx - 5y = 3$ is perpendicular to the line $3x - 2y = 7$.

24. Find the linear function f such that $f(2) = 3$ and $f(-1) = 1$.

25. Find the linear function f such that $f(-5) = 1$ and $f(0) = 4$.

26. Find $f(-3)$ if the linear function f satisfy $f(-6) = -1$ and $f(-2) = -5$.

27. The following table show y, the weight required to stretch a spring in different distances x; where the weight is measured in kg and the distance in cm.

Distance: x	3	5	6	8
Weight: y	14	19	21.5	26.5

(a) Is the function $y = f(x)$ a linear function?

(b) Find $f(2)$.

(c) Find x if $f(x) = 17.5$.

28. The table below shows the final grade, y, of the students of a math course based on their term work ,x. Fill up the table assuming that the final grade of each student is related by a linear function to his/her term work mark.

Term work: x	45	50	70	
Final grade: y	34		64	82

29. It costs \$350 to print 100 copies of a manual and \$800 to print 250 copies.

(a) Express the cost, y, in a linear equation with the number of copies, x.

(b) How much would it cost to print 650 copies?

30. If y represents the amount of water which evaporates from a swimming pool in summer and x represents the surface area of the pool, then express y as a linear function of x where 24 and 30 gallons of water evaporated from two swimming pools of surfaces 120 and 150 square feet respectively. How many gallons of water would evaporate if the surface of a swimming pool is 210 square feet? What is the surface area of a swimming pool if 21 gallons of water evaporates from it during summer?

31. In a factory, it costs \$2.25 per key chain to be produced and the daily fixed cost is \$755. Express the total daily cost $C(x)$ in terms of the number of key chains produced, x. Then find the total cost if 1027 key chains are produced in one day.

32. In a small city, the taxi fare y in \$ is a linear function of the distance x moved in km. Write the linear equation between y and x and then fill up the following table.

distance x in km	0	5	8	10	
taxi fare y in \$		9.50		15.75	25.75

33. The population, y, of a small city has been growing linearly since 1985 as follows:

$$y = 2755x + 12000$$

where x is years since 1985.

(a) What was the population in 1985?

(b) What was the population in 2000?

(c) When will the population reach 100,000?

3.4 Final Answers to Exercises

Section 3.1

1. (a) $\sqrt{52}$
   (b) $\sqrt{13}$
   (c) $10\sqrt{2}$
   (d) 5
   (e) $\sqrt{5}$
   (f) $\frac{\sqrt{193}}{6}$
   (g) $\sqrt{6}$
   (h) $2\sqrt{3}$
   (i) $2\sqrt{a^2+b^2}$

2. (a) $(1,4)$
   (b) $(-\frac{9}{2},\frac{3}{2})$
   (c) $(-2,\frac{7}{2})$
   (d) $(-\frac{7}{2},\frac{5}{4})$
   (e) $(\frac{19}{12},-\frac{5}{4})$
   (f) $(\frac{5}{2}\sqrt{3},0)$
   (g) $(2\sqrt{2},-1)$
   (h) (a,b)

3. $(3,2)$

4. $(-\frac{20}{3},-\frac{3}{2})$

5. $5+3\sqrt{5}+2\sqrt{10}$

6. 28

7. $6/5$

8. 45π

9. $AB=AC=\sqrt{17}$

10. $(0,1)$ and $(0,9)$

11. $(5/4,0)$

12. $y=-1$

13. $x=-1,\ 3$

14. $y=3,\ 13$

15. Note $AB = \sqrt{65}$, $AC = \sqrt{20}$, $BC = \sqrt{45}$, and $AB^2 = AC^2 + BC^2$.

Section 3.2

1. (a) Yes (b) Yes (c) Yes (d) No (e) Yes
2. (a) Yes (b) No (c) Yes (d) Yes (e) Yes (f) No
3. (a) Yes (b) No (c) Yes (d) Yes (e) No (f) No (g) Yes (h) No (i) Yes
4. (a) $D = \{-7, -1, 0, 2, 3\}$ and $R = \{0, 2, 3, 4\}$
   (b) $D = \{-3, -2, 0, 3, 5\}$ and $R = \{-3\}$
   (c) $D = R = \mathbb{Z}$
5. (a) -3 (b) 2 (c) 0 (d) $x = -3$
6. (a) 2 (b) 32 (c) 1 (d) $x = 19$
7. (a) 2 (b) 2 (c) 8 (d) $n = 20$
8. (a) $D_g = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ and $R_g = \{1, 2, 4, 8, 16, 32, 64, 128, 256\}$
   (b) $D_h = \{0, 1, 3, 7, 9, 11, 14, 20\}$ and $R_h = \{1, 2, 6, 7, 8\}$
9. (a) 9, 7, 5, and -1 respectively
   (b) 24, 13, 8, and 29 respectively
   (c) 4, -1, 0, and -21 respectively
   (d) -4, -6, -6, and 6 respectively
   (e) 4/5, 3/4, 2/3 $<$ and undefined respectively
   (f) 1/4, 1/2, 1, and 8 respectively
   (g) -3, -2, $\sqrt{2} - 3$, and $\sqrt{5} - 3$ respectively

(h) undefined, undefined, undefined and 1 respectively

(i) $-5/7, \ -3/4, \ -1/3$ and $5/12$ respectively

(j) $1/27, \ 1/9, \ 1/3$ and 9 respectively

10. (a) $(-1,-1)$ and $(1/3,3)$
    (b) $(-2,11)$ both
    (c) $(2,2)$, $(0,0)$ and $(1,0)$
    (d) $(3,0)$ and $(12,3)$

11. (a) 0 (b) 1 (c) $x=0$

12. (a) -1 (b) 2 (c) $x=0, \ 4$

13. (a) 1 (b) $4/5$ (c) $x=0$

14. (a) -23 (b) 75 (c) -16 (d) $x=6/7$

15. (a) -2 (b) -4 (c) 136 (d) $x=-2, \ 1/3$

16. (a) 9 (b) 0 (c) 8 (d) 16 (e) 8 (f) 9 (g) 1

17. (a) 14 (b) $-3a+5$ (c) $-3a-1$ (d) $-3a+4$

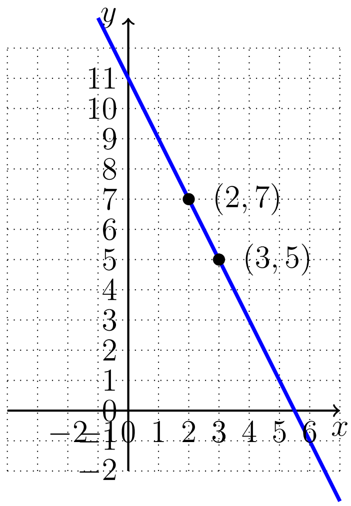
18. (a) $-1/2$ (b) $x/2$ (c) $(x-2)/2$ (d) $(x-1)/2$

19. (a) $\dfrac{x}{x+3}$ (b) $\dfrac{1-2x}{1+x}$ (c) $\dfrac{2x-1}{x+1}$ (d) $\dfrac{x-5}{2(x+1)}$

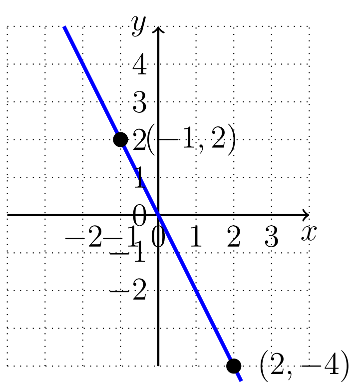
20. (a) $\mathbb{R}$
    (b) $\mathbb{R}$
    (c) $\mathbb{R}$
    (d) $\mathbb{R}$
    (e) $\mathbb{R}\setminus\{3\}$
    (f) $\mathbb{R}\setminus\{-1/2\}$
    (g) $[1,\infty)$
    (h) $(-\infty,2]$
    (i) $[0,1)\cup(1,\infty)$
    (j) $(1,\infty)$
    (k) $(-2,\infty)$

Section 3.3

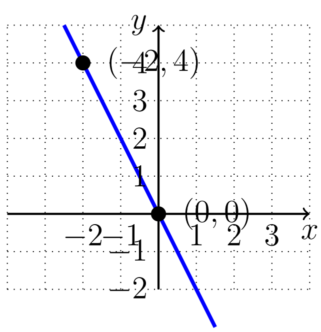
1. (a) $m = -2, \quad y = -2x + 11$



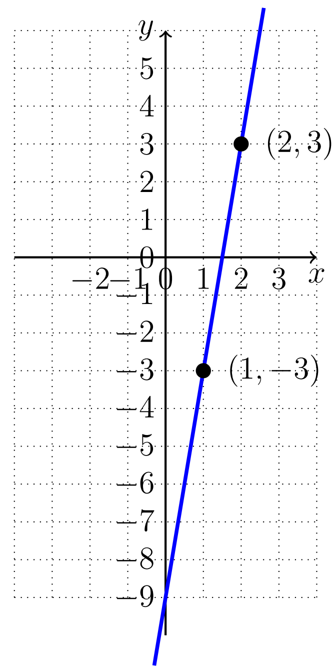
(b) $m = -2, \quad y = -2x$



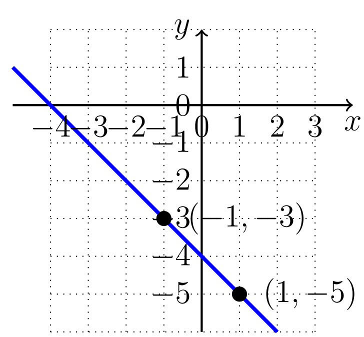
(c) $m = -2, \quad y = -2x$



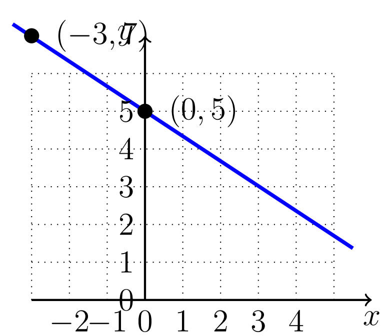
(d) $m = 6, \quad y = 6x - 9$



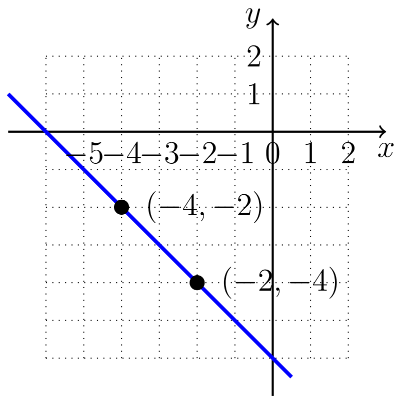
(e) $m = -1, \quad y = -x - 4$



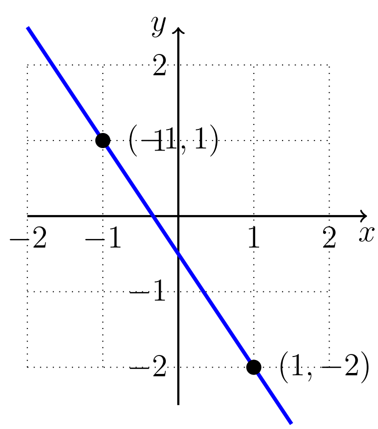
(f) $m = \dfrac{-2}{3}, \; y = \dfrac{-2}{3}x + 5, \; or$
$3y = -2x + 15$



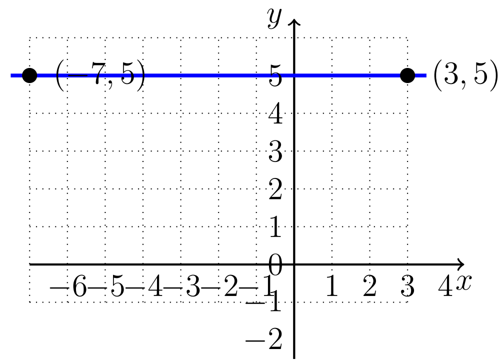
(g) $m = -1, \quad y = -x - 6$



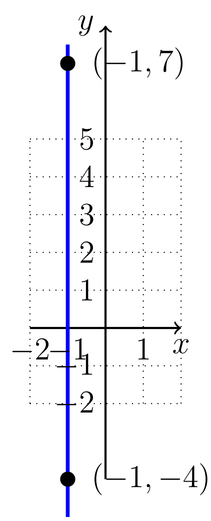
(h) $m = \dfrac{-3}{2}, \quad y = \dfrac{-3}{2}x - \dfrac{1}{2}, \quad$ *or* $2y = -3x - 1$



(i) $m = 0, \quad y = 5$



(j) m is undefined, $x = -1$



2. $y = -2x + 3$

3. $y = \dfrac{-1}{3}x - \dfrac{1}{3}, \quad$ or $\quad 3y = -x - 1$

4. $y = \dfrac{2}{5}x - \dfrac{14}{5}, \quad$ or $\quad 5y = 2x - 14$

5. $y = \dfrac{5}{3}x + \dfrac{17}{3}, \quad$ or $\quad 3y = 5x + 17$

6. $y = \dfrac{-7}{3}, \quad$ or $\quad 3y = -7x$

7. $y = \frac{1}{2}x - 2, \quad \text{or} \quad 2y = x - 4$

8. **(b)** and **(c)**

9. **(b)**, **(c)** and **(d)**

10. $y = -3x - 5$

11. $y = \frac{2}{3}x - 3$

12. $y = -4$

13. $x = 3$

14. $y = \frac{1}{2}x - \frac{5}{2}, \quad \text{or} \quad 2y = x - 5$

15. $y = \frac{-2}{3}x - \frac{14}{3}, \quad \text{or} \quad 3y = -2x - 14$

16. $x = -3$

17. $y = -1$

18. $m_1 = m_2 = \frac{3}{2}$

19. $m_1 = \frac{3}{2}, \quad m_2 = \frac{-2}{3}$

20. $k = -4$

21. $n = -14$

22. $a = 6$

23. $k = \frac{-10}{3}$

24. $y = \frac{2}{3}x + \frac{5}{3}, \quad or \quad 3y = 2x + 5$

25. $y = \frac{3}{5}x + 4, \quad or \quad 5y = 3x + 20$

26. $f(-3) = -4$

27. **(a)** Yes, it is, **(b)** $f(2) = 11.5$, **(c)** $x = 4.6$

28. $f(50) = 40, \quad f(85) = 82$

29. **(a)** $y = 3x + 50$, **(b)** $y = \$2000$

30. $y = f(x) = \dfrac{x}{5}, \quad f(210) = 42, \quad f(105) = 21$

31. $C(x) = 2.25x + 755, \quad C(1027) = 3065.75$

32. $y = f(x) = 1.25x + 3.25, \quad f(0) = 3.25, \quad f(8) = 13.25, \quad f(18) = 25.75$

33. **(a)** 12000, **(b)** 53325, **(c)** almost 32 years.

Chapter 4
Linear Inequalities

4.1 Solving Linear Inequalities

A linear inequality is obtained if in a linear equation the sign $=$ is replaced by any one of the following

$$<, \qquad \leq, \qquad >, \qquad \geq .$$

Here are some examples:

$$\begin{aligned} x+2 &> 0, \\ 8x+1 &\geq 5x-2, \\ 10(x-1) &< 5-(2x+3), \\ 2(x-3)-5 &\leq 3(x+2)-18. \end{aligned}$$

To solve a linear inequality, we isolate the variable—as we do in the case of a linear equation—by performing the same operations on each side of the inequality, **except that we should reverse the inequality sign (e.g., $<$ becomes $>$, $\leq$ becomes $\geq$, etc.) whenever we either multiply or divide the two sides of the inequality by a negative number.**

Examples

(1) Solve the inequality $\qquad 5x+2>22.$

Solution.

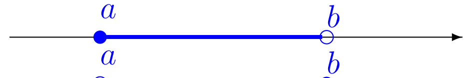
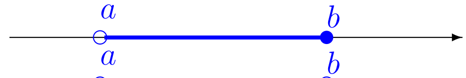
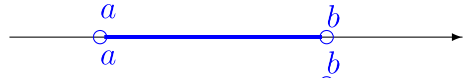
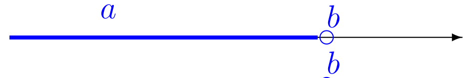
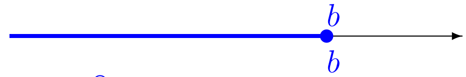
$$\begin{aligned} 5x+2>22 &\implies 5x>20 \\ &\implies x>4. \end{aligned}$$

Remark/Notation. It is customary in this topic to use "**interval**" notations for the solutions of inequalities. The interval notations are defined as follows:

Inequality	Solution Set/Interval
$a \le x \le b$	$[a, b]$
$a \le x < b$	$[a, b)$
$a < x \le b$	$(a, b]$
$a < x < b$	(a, b)
$x < b$	$(-\infty, b)$
$x \le b$	$(-\infty, b]$
$x > a$	$(a, +\infty)$
$x \ge a$	$[a, +\infty)$

So the solution to **(1)** can be given in interval notation as $(4, +\infty)$.

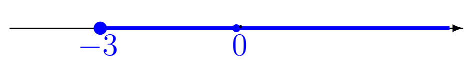
The geometric interpretation of the solutions of inequalities are given as following solution graphs.

Inequality	Solution Graph
$a \le x \le b$	
$a \le x < b$	
$a < x \le b$	
$a < x < b$	
$x < b$	
$x \le b$	
$x > a$	
$x \ge a$	

(2) Solve the inequality $\quad 2(x-3) \le 6(x+1)$.
Solution.

$$\begin{aligned} 2x - 6 \le 6x + 6 &\implies -4x \le 12 \\ &\implies x \ge -3. \end{aligned}$$

So, the solution set is $[-3, +\infty)$ and the solution graph is given by



(3) Solve the inequality $\quad 7(5x-2)-9(3x+1) > 5(3x-8)-53.$
Solution.

$$\begin{aligned} 35x-14-27x-9 > 15x-40-53 &\implies -7x > -70 \\ &\implies x < 10. \end{aligned}$$

So, the solution set would be $(-\infty, 10)$ and the solution graph is

(4) Solve the "double" inequality $\quad -4 < 3x-7 \leq 3.$
Solution.

$$\begin{aligned} & -4 < 3x-7 \leq 3 \\ \implies & -4+7 < 3x \leq 3+7 \\ \implies & 3 < 3x \leq 10 \\ \implies & 1 < x \leq 10/3. \end{aligned}$$

and the solution set is $(1, 10/3]$ and the solution graph is

(5) Solve the inequality $\quad 8 \leq 3 - \frac{5}{2}x < 18.$
Solution.

$$\begin{aligned} & 8 \leq 3 - \frac{5}{2}x < 18 \\ \implies & 16 \leq 6 - 5x < 36 \\ \implies & 10 \leq -5x < 30 \\ \implies & -2 \geq x > -6 \quad \text{or equivalently} \quad -6 < x \leq -2. \end{aligned}$$

Thus, the solution set is $(-6, -2]$ and the solution graph is as follows

(6) Solve the inequality $\dfrac{x+3}{x} \leq 9.$

Solution.

In this example, first we need to deal with the x in the denominator. So we consider the following two cases:

Case I. If $x > 0$ then:

$$\begin{aligned} \frac{x+3}{x} \leq 9 &\implies x+3 \leq 9x \\ &\implies 3 \leq 8x \\ &\implies 3/8 \leq x \end{aligned}$$

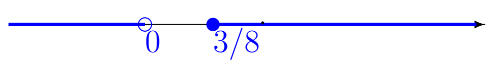
Since $0 < x$ **and** $\dfrac{3}{8} \leq x$ so in this case we have $\dfrac{3}{8} \leq x$.

Case II. If $x < 0$ then:

$$\begin{aligned} \frac{x+3}{x} \leq 9 &\implies x+3 \geq 9x \\ &\implies 3 \geq 8x \\ &\implies 3/8 \geq x \end{aligned}$$

For this case, the two conditions $x < 0$ **and** $x \leq \frac{3}{8}$ yields to $x < 0$.

Therefore, the final answer is $x < 0$ or $\dfrac{3}{8} \leq x$ and the solution graph is given by



Exercises

1. Solve each inequality and give the final answer using inequalities, intervals and on the real line.

   (a) $9x + 13 \geq 8x$

   (b) $5x + 7 < 2x + 1$

   (c) $\frac{5}{3}(x+1) \leq -x + \frac{2}{3}$

   (d) $3x + 7 \leq 4x - 2$

   (e) $9 - 2x \geq 24 + 3x$

   (f) $9(x-1) > 13 + 7x$

   (g) $2(x-3) - 5 \leq 3(x+2) - 18$

   (h) $3(2-x) > x + 52$

   (i) $2x - 3(x+1) > 4(x-3) + 9$

   (j) $2(x+1) + 2x - 32 \geq x - 6$

   (k) $0.10(18 + x) \leq 0.25x - 1.2$

   (l) $3 < x - 2 < 7$

   (m) $8 \geq 2x + 5 > -1$

   (n) $0 < 10 - 5x \leq 11$

   (o) $4 \leq 7x + 3 < 24$

   (p) $5 \leq 1 - 2x \leq 10$

   (q) $8 > 2 + \frac{1}{2}x \geq -1$

   (r) $3 < \frac{5-3x}{2} \leq 7$

   (s) $-3 < 12x - 2 \leq 5$

4.2 Absolute Value Linear Inequalities

Many important uses of inequalities involve **absolute values**. The absolute value of x denoted by $|x|$ is the magnitude of the real number x without regard to its sign and mathematically it is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

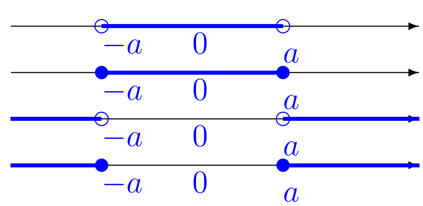
Note that if x is a negative real number, then $|x| = -x$ is positive and this is exactly what we want. Therefore $|5| = 5$, and $|-5| = 5$ as well.

In the following table we will consider four possible scenarios that may happen for linear inequalities involving absolute values. Note that a is a positive real number.

Absolute Value Inequality	Inequality	Solution Set/Interval
$\lvert x\rvert < a$	$-a < x < a$	$(-a, a)$
$\lvert x\rvert \leq a$	$-a \leq x \leq a$	$[-a, a]$
$\lvert x\rvert > a$	$x < -a$ or $x > a$	$(-\infty, -a) \cup (a, \infty)$
$\lvert x\rvert \geq a$	$x \leq -a$ or $x \geq a$	$(-\infty, -a] \cup [a, \infty)$

In the following table, we sketch a graph for each of the four cases.

Ab. V. Inequality	Inequality	Solution graph
$\lvert x\rvert < a$	$-a < x < a$	
$\lvert x\rvert \leq a$	$-a \leq x \leq a$	
$\lvert x\rvert > a$	$x < -a$ or $x > a$	
$\lvert x\rvert \geq a$	$x \leq -a$ or $x \geq a$	



Examples

(1) Solve the inequality $|x - 2| < 5$.

Solution.

$$\begin{aligned} |x-2| < 5 &\implies -5 < x - 2 < 5 \\ &\implies -3 < x < 7. \end{aligned}$$

(3) Solve the inequality $\quad |2x+3| \geq 10.$
Solution.

$$\begin{aligned} |2x+3| \geq 10 &\implies 2x+3 \leq -10 \text{ or } 2x+3 \geq 10 \\ &\implies 2x \leq -13 \text{ or } 2x \geq 7 \\ &\implies x \leq \frac{-13}{2} \text{ or } x \geq \frac{7}{2} \end{aligned}$$

(2) Solve the inequality $\quad |2-4x| \leq 8.$
Solution.

$$\begin{aligned} |2-4x| \leq 8 &\implies -8 \leq 2-4x \leq 8 \\ &\implies -10 \leq -4x \leq 6 \\ \text{Divide them by } -4 \text{ andchange thedirection of inequalities} \\ &\implies \frac{5}{2} \geq x \geq \frac{-3}{2} \end{aligned}$$

The final answer is better to be written as $\dfrac{-3}{2} \leq x \leq \dfrac{5}{2}.$

Exercises

1. Solve each inequality and give the final answer using inequalities.

   (a) $|5x| < 10$

   (b) $|4x - 1| \leq 3$

   (c) $|7 - 2x| \geq 1$

   (d) $5 < 2 + |2x - 3|$

   (e) $3 + |4 - x| > 10$

   (f) $\dfrac{|2x - 5|}{3} \geq 7$

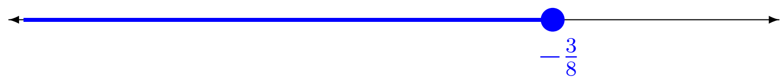
   (g) $\left|1 - \dfrac{2x}{3}\right| < 1$

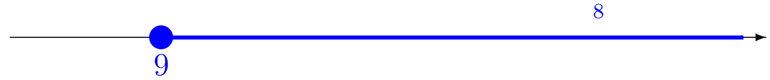
4.3 Final Answers to Exercises

Section 4.1

1. (a) $x \geq -13$; $[-13, \infty)$;

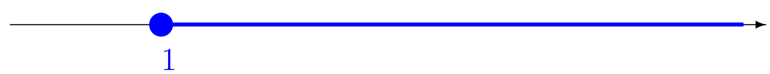

(b) $x < -2$; $(-\infty, -2)$;

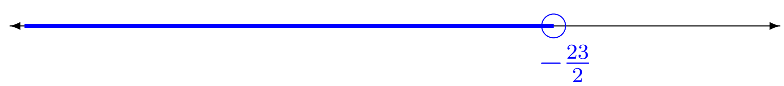

(c) $x \leq -\frac{3}{8}$; $(-\infty, -\frac{3}{8}]$;


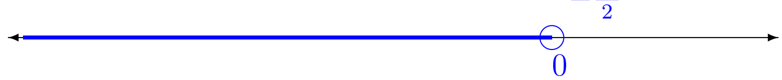
(d) $x \geq 9$; $[9, \infty)$;


(e) $x \leq -3$; $(-\infty, -3]$;


(f) $x > 11$; $(11, \infty)$;

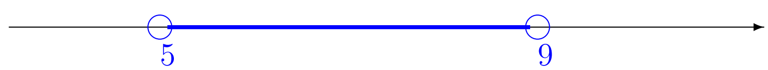

(g) $x \geq 1$; $[1, \infty)$;


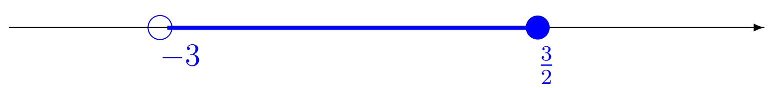
(h) $x < \frac{-23}{2}$; $(-\infty, -\frac{23}{2})$;


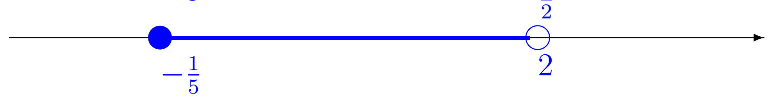
(i) $x < 0$; $(-\infty, 0)$;


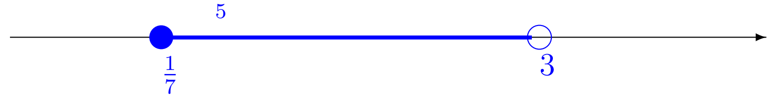
(j) $x \geq 8$; $[8, \infty)$;

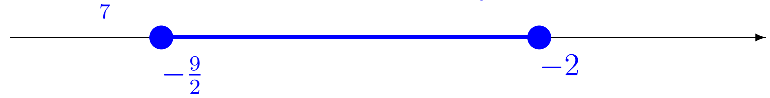

(k) $x > 20$; $(20, \infty)$;

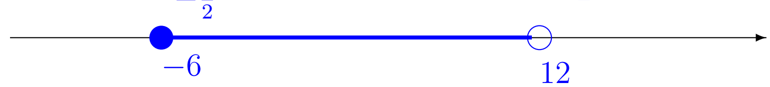

(l) $5 < x < 9$; $(5, 9)$;


(m) $-3 < x \leq \frac{3}{2}$; $(-3, \frac{3}{2}]$;


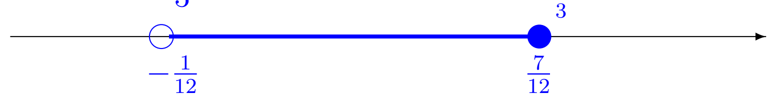
(n) $-\frac{1}{5} \leq x < 2$; $[-\frac{1}{5}, 2)$;


(o) $\frac{1}{7} \leq x < 3$; $[\frac{1}{7}, 3)$;


(p) $-\frac{9}{2} \leq x \leq -2$; $[\frac{-9}{2}, -2]$;


(q) $-6 \leq x < 12$; $[-6, 12)$;


(r) $-3 < x < -\frac{1}{3}$; $(-3, -\frac{1}{3})$;


(s) $-\frac{1}{12} < x \leq \frac{7}{12}$; $(-\frac{1}{12}, \frac{7}{12}]$;


Section 4.2

1. (a) $-2 < x < 2$

   (b) $\dfrac{-1}{2} \leq x \leq 1$

   (c) $x \leq 3$ or $x \geq 4$

   (d) $x < 0$ or $x > 3$

   (e) $x < -3$ or $x > 11$

   (f) $x \leq -8$ or $x \geq 13$

   (g) $0 < x < 3$

Chapter 5
System of Linear Equations

5.1 Systems of Two Linear Equations

By a system of (two) linear equations we mean a pair of equations in two variables, x and y say, of the form

$$\begin{cases} ax + by = c, \\ a'x + b'y = c'. \end{cases}$$

To solve the system means to find a pair of values, one value for x and another one for y, which simultaneously satisfies both equations. For instance the solution to the system

$$\begin{cases} 2x + 3y = 7, \\ 5x - 4y = 6, \end{cases}$$

is $x = 2,\ y = 1$ (verify this to yourself!)

Geometric Interpretation of Solution(s)

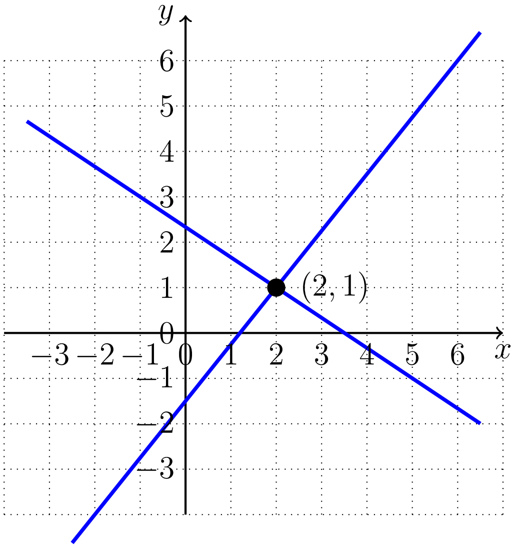
If a system of two linear equations has a unique solution, the solution can be seen as the point (x, y) of the intersection of the lines $ax + by = c$ and $a'x + b'y = c'$. Alternatively the system may have no solution or infinitely many solutions. Systems that have solutions are called **consistent** and those without any solutions are **inconsistent**. Let us now, take a closer look at the three possible scenarios and interpret them geometrically.

Case I. Let two lines have different slopes and therefore intersect each other at a single point which is the solution of the given system.

Example.

(1) Sketch the graph of the lines and find the solution(s) if exists geometrically.

$$\begin{cases} 2x + 3y = 7, \\ 5x - 4y = 6, \end{cases}$$



This system has a **unique solution** $(2, 1)$ or equivalently $x = 2$ and $y = 1$.

Case II. If two lines are parallel (or have the same slope but different y-intercepts), then they do not intersect each other and the system is called **inconsistent.**

Example.

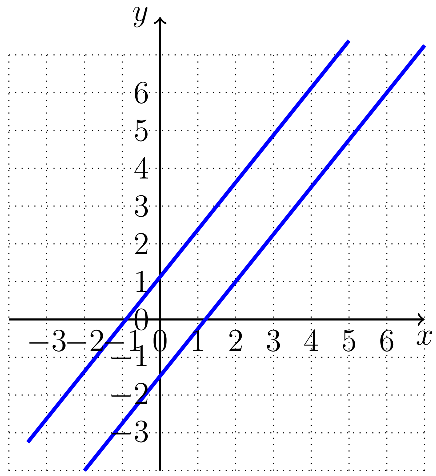
(2) Sketch the graph of the lines and find the solution(s) if exists geometrically.

$$\begin{cases} 10x - 8y = -9, \\ 5x - 4y = 6, \end{cases}$$

It is easily seen that this system is equivalent to the system:

$$\begin{cases} 5x - 4y = -9/2, \\ 5x - 4y = 6, \end{cases}$$

Note that one algebraic expression cannot give different results. Therefore, such a system can not have any solutions. Note also that the lines have the same slope $m = \frac{5}{4}$ but different y-intercepts.



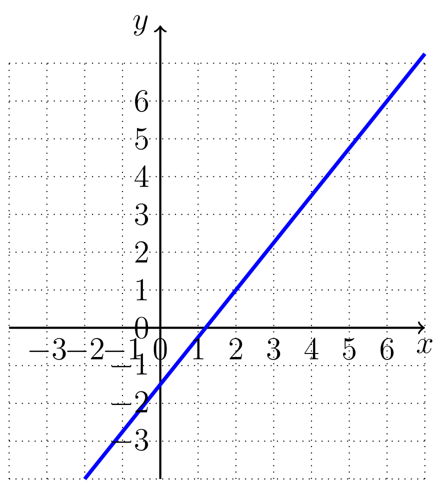
Case III. And the last case happens when the two lines are overlapping. In this case the system has infinitely many solutions since each point on these lines is a solution of the system.

Example.

(3) Sketch the graph of the lines and find the solution(s) if exists geometrically.

$$\begin{cases} 10x - 8y = 12, \\ 5x - 4y = 6, \end{cases}$$

Note that if we multiply both sides of the second equation by 2, it will become the same as the first equation. This means that the two equations in the system are equivalent. As a result we have only one line and every point on this line is a solution of the system.



In the following section, you will learn how to algebraically solve a linear system of two equations using **Substitution Method** and **Elimination Method.**

5.2 Solving Systems of Two Linear Equations

How to solve systems:

In these notes we shall discuss two main techniques to solve systems of equations: Substitution and Elimination.

Substitution Method

The idea is to express (=to find) one of the variables in terms of the other one and "substitute" the expression found in the other equation. We shall now demonstrate this in a few examples.

Examples

(1) Solve the system:

$$\begin{cases} 2x + 3y = 7, \\ 5x - 4y = 6. \end{cases}$$

Solution. If from the second equation we isolate y, we can expressed it in terms of x as

$$y = \frac{5}{4}x - \frac{3}{2}.$$

Now, we substitute this for y in the first equation and obtain

$$2x + 3(\frac{5}{4}x - \frac{3}{2}) = 7$$

which is just a linear equation in one variable (x in this case), hence very easy to solve:

$$2x + \frac{15}{4}x - \frac{9}{2} = 7 \implies 23x = 46 \implies x = 46/23 = 2.$$

Now by putting $x = 2$ in any one of the equations, we find the value of y:

$$2 \cdot 2 + 3y = 7 \implies 3y = 3 \implies y = 1.$$

Hence $x = 2,\ y = 1$ is the solution to the system.

Remark. If any one of the variables appears with the coefficient ± 1, it is recommended to isolate that variable as it will lead to simpler computations.

(2) Solve the system:

$$\begin{cases} x + 5y = 13, \\ -3x + 2y = 12. \end{cases}$$

Solution. From the first equation we get $x = 13 - 5y$; substituting this in the other equation and simplifying it yields the linear equation $17y - 39 = 12$ whose solution is easily found as $y = 3$. Now plugging this value for y in the first equation and solving for x yields $x = -2$.

Elimination Method

In this technique, one eliminates one of the variables by either adding or subtracting the sides of the two equations, provided the two coefficients of the variable under elimination are either equal or opposite. If the coefficients of neither of the variables happens to be equal or opposite, we will make them so by multiplying our equations through by suitable numbers.

(3) Solve by elimination:

$$\begin{cases} 4x + 3y = 19, \\ 7x - 3y = -8. \end{cases}$$

Solution. As the coefficients of y are 3 and -3 respectively, we can easily eliminate y (simply by adding up the left and the right sides of the equations) and get

$$(4x + \cancel{3y}) + (7x - \cancel{3y}) = 19 + (-8) \implies 11x = 11 \implies x = 1.$$

This in turn determines the value of y as $y = 5$.

(4) Solve by eliminating x:

$$\begin{cases} 3x + 2y = 26, \\ 4x - 5y = 50. \end{cases}$$

Solution. To eliminate x it is necessary for the two coefficients of x to be either equal or opposite, which is not the case here. However, we can make that happen if we multiply the first equation (both sides obviously) by 4 say and the second equation by -3; then the new coefficients of x will be 12 and -12 respectively, so that if we add the new equations

obtained x will get eliminated:

$$\begin{cases} 4 \times (3x + 2y) = 4 \times 26, \\ (-3) \times (4x - 5y) = (-3) \times 50 \end{cases} \Longrightarrow \begin{cases} 12x + 8y = 104, \\ -12x + 15y = -150 \end{cases} \Longrightarrow$$

$$\overset{\text{adding}}{\Longrightarrow} \Big(\cancel{12x} + 8y\Big) + \Big(-\cancel{12x} + 15y\Big) = 104 + (-150) \Longrightarrow 23y = -46 \Longrightarrow$$
$$\Longrightarrow y = -2 \Longrightarrow x = 10.$$

(5) (A Word Problem) If two CD's and three tapes cost \$36 and five CD's and four tapes cost \$76, find the price of each.

Solution. Solving this word problem amounts to solving the following system

$$\begin{cases} 2x + 3y = 36, \\ 5x + 4y = 76. \end{cases}$$

where x (resp. y) is the cost of each CD (resp. tape). Using either of methods explained above one finds the solution to the system as $x = 12$ (i.e., a CD costs \$12) and $y = 4$ (i.e., a tape costs \$4).

Exercises

1. Solve the systems.

(a) $\begin{cases} x+y=4 \\ x-y=6 \end{cases}$

(b) $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

(c) $\begin{cases} x+y=4 \\ 2x+y=5 \end{cases}$

(d) $\begin{cases} y=x-2 \\ y=-x+6 \end{cases}$

(e) $\begin{cases} y=5x-8 \\ 7x-y=10 \end{cases}$

(f) $\begin{cases} x-2y=-11 \\ -x+5y=26 \end{cases}$

(g) $\begin{cases} 2x+3y=8 \\ 4x-3y=4 \end{cases}$

(h) $\begin{cases} 2x-3y=1 \\ 4x-6y=2 \end{cases}$

(i) $\begin{cases} 5x+7y=10 \\ 3x-14y=6 \end{cases}$

(j) $\begin{cases} 5x+2y=0 \\ 3x+5y=0 \end{cases}$

(k) $\begin{cases} 7x+10y=13 \\ 4x+5y=6 \end{cases}$

(l) $\begin{cases} 3x-6y=-1 \\ 6x-4y=2 \end{cases}$

(m) $\begin{cases} 2x-3y=16 \\ 3x+4y=7 \end{cases}$

(n) $\begin{cases} x-2y=6 \\ -2x+4y=13 \end{cases}$

(o) $\begin{cases} 5x+2y=1 \\ 2x+3y=-15 \end{cases}$

(p) $\begin{cases} 3x+5y=-18 \\ 4x+2y=-10 \end{cases}$

(q) $\begin{cases} 5x+15y=11 \\ 2x+6y=-3 \end{cases}$

(r) $\begin{cases} 3x-2y=27 \\ 2x+5y=-1 \end{cases}$

(s) $\begin{cases} 2x+5y=2 \\ 3x+3y=1 \end{cases}$

(t) $\begin{cases} 2x-5y=4 \\ 3x-2y=4 \end{cases}$

(u) $\begin{cases} 2x-3y=-16 \\ y=-2x \end{cases}$

(v) $\begin{cases} 9x-y=25, \\ 2y=4-9x \end{cases}$

(w) $\begin{cases} 2x+9y=16 \\ 5x=1-3y \end{cases}$

(x) $\begin{cases} 0.2x-0.3y=-0.95 \\ 0.4x-0.1y=0.55 \end{cases}$

(y) $\begin{cases} 0.08x-0.04y=-0.11 \\ 0.02x-0.06y=-0.09 \end{cases}$

2. The sum of two numbers is 111 and their difference is 45. Find them.

3. Find two numbers whose sum is 72 and one of them is twice the other one.

4. Find two numbers whose sum is 18 and 3 times the smaller one is 10 more then the larger one.

5. The sum of a larger number and twice of a smaller number is 87. Find them if their difference is 36.

6. Find two numbers whose sum is 98 such that twice of one of them minus the half of the other one is 56.

7. Find two supplementary angles such that the measure of the larger angle is $15°$ more than twice the measure of the smaller one.

8. Find two complementary angles such that one third of the measure of the larger one minus double the measure of the smaller angle is $16°$.

9. There are 8173 students in a college such that the number of girls is 421 more than the number of boys. How many of each are there?

10. A bookstore sold 18 books for a total of $176. If some of the books were sold for $7 and some for $12, how many of each were sold?

11. If a man has $315 in ten and five dollar bills, how many of each does he have if he has 42 bills in all?

12. If 100 coins consist of quarters and loonies, how many of each are there if they are $79.75 in total?

13. If 100 coins consists of quarters and dimes, how many of each are there if they are $20.80 in total?

14. A computer online service charges one hourly rate for regular use and a higher hourly rate for premium service. If a customer is charged $14 for 9 hours of basic and 2 hours of premium use and another customer is charged $13.50 for 6 hours of regular use and 3 hours of premium use, how much is the service charge per hour for regular and premium services?

15. A company ordered 4 turkey sandwiches and 7 french fries for a total cost of $38.30 before tax. The next day they ordered 5 turkey sandwiches and 5 french fries for a total of $40.75 before tax. What are the prices for a turkey sandwich and a french fries?

16. A baker purchased 12 lb of wheat flour and 15 lb of rye flour for a total cost of \$18.30. A second purchase at the same price included 15 lb of wheat flour and 10 lb of rye flour for a total of \$16.75. Find the cost per pound of each flour.

17. For a club trip, 4 members and 3 non-members must pay a total of \$159, while 3 members and 2 non-members must pay \$112. What is the price for each?

5.3 Final Answer to Exercises

Section 5.2

1. (a) $x = 5,\ y = -1$
   (b) $x = 2,\ y = -1$
   (c) $x = 1,\ y = 3$
   (d) $x = 4,\ y = 2$
   (e) $x = 1,\ y = -3$
   (f) $x = -1,\ y = 5$
   (g) $x = 2,\ y = \frac{4}{3}$
   (h) Infinitely many Solutions.
   (i) $x = 2,\ y = 0$
   (j) $x = 0,\ y = 0$
   (k) $x = -1,\ y = 2$
   (l) $x = \frac{2}{3},\ y = \frac{1}{2}$
   (m) $x = 5,\ y = -2$
   (n) No Solution.
   (o) $x = 3,\ y = -7$
   (p) $x = -1,\ y = -3$
   (q) No Solution.
   (r) $x = 7,\ y = -3$
   (s) $x = -\frac{1}{9},\ y = \frac{4}{9}$
   (t) $x = \frac{12}{11},\ y = \frac{-4}{11}$
   (u) $x = -2,\ y = 4$
   (v) $x = 2,\ y = -7$
   (w) $x = -1,\ y = 2$
   (x) $x = 2.6,\ y = 4.9$
   (y) $x = -0.75,\ y = 1.25$

2. 78 and 33
3. 24 and 48
4. 11 and 7
5. 53 and 17
6. 56 and 42
7. 125° and 55°
8. 84° and 6°
9. 3876 boys and 4297 girls.
10. 10 ($12) and 8 ($7)
11. 21 ($10) and 21 ($5)

12. 73 loonies and 27 quarters.

13. 72 quarters and 28 dimes.

14. \$1 regular and \$2.50 premium.

15. \$6.25 turkey sandwich and \$1.90 french fries.

16. \$0.65 wheat flour and \$0.70 rye flour.

17. \$18 members and \$29 non-members.

Chapter 6

Percentages and Ratios

6.1 Percentages

The word **percent** means ”per hundred”, and basically, $Y\%$ of a value X is the same as

$$Y\% \cdot X = \frac{Y}{100} \cdot X.$$

If X is 100 (or any multiple of 100) then the result can be obtained right away. For instance, if we pay 15% tax on our \$100 bill, then we need to pay \$15 tax. For the other values of X however, it only needs a multiplication.

Let us look at some problems regarding percentages.

Examples

(1) A store is giving a discount of 20% on all merchandises. If the reduced price of an item is \$130, find its original price.

Solution. Let us consider $\$X$ for the original price. So, $80\% \cdot X = 0.80 \cdot X = 130$ and therefore $X = \dfrac{130}{0.8} = \$162.50$.

(2) In a small city of $753,000$ people, 22% have an income of at least $\$90,000$, of which 40% have higher than $\$120,000$ annual income. How many people have an income of higher than $\$120,000$?

Solution. First, we find the number of people with the income higher than $\$90,000$ as:

$$22\% \cdot 753,000 = \frac{22}{100} \cdot 753,000 = 0.22 \cdot 753,000 = 165,660$$

and the number of people having income greater than $\$120,000$ will be:

$$40\% \cdot 165,660 = \frac{40}{100} \cdot 165,660 = 0.40 \cdot 165,660 = 66,264.$$

(3) Find the original price of an item, if it costs \$149.50 after 15% taxes?

Solution. Let us assume that $\$X$ is the original price. Then

$$X + 15\% \cdot X = 149.50 \implies X + 0.15 \cdot X = 149.50 \implies 1.15 \cdot X = 149.50$$

So, $X = \dfrac{149.50}{1.15} = \$130.$

(4) In Quebec, we pay 5% for federal tax and 9.975% for provincial taxes on most of goods and services. Find the listed price of a TV set if one gets a bill of \$2857?

Solution. Similar to the previous problem, we assume the listed price is $\$X$. Then

$$\begin{aligned} X + 5\% \cdot X + 9.975\% \cdot X = 2857 &\Longrightarrow X + 0.05 \cdot X + 0.09975 \cdot X = 2857 \\ &\Longrightarrow 1.14975 \cdot X = 2857 \end{aligned}$$

Therefore, the price of the TV set was $X = \dfrac{2857}{1.14975} \approx \$2485$

(5) The federal income tax follows the following table:

annual incomes	percentage
$\$0 - \$12,500$	0%
$\$12,501 - \$49,000$	15%
$\$49,001 - \$98,000$	20%
$\$98,001 - \$152,00$	26%

(a) Calculate your federal tax if you have $\$82,500$ annual income.

(b) Can you find your friend's salary, if you know he pays almost $\$11,000$ for his federal tax.

Solution.

(a) Since $\$82,500$ is in the third bracket (third row of the table), we will do as follows:

$$20\%\cdot(82500-49000)+15\%\cdot(49000-12500) = 0.20\cdot 33500+0.15\cdot 36500 = 6700+5475 = \$12175$$

(b) Having the answer of part **(a)**, gives us a hint of having the salary $\$X$ of our friend in the same bracket namely between $\$49,001$ to $\$98,000$. So,

$$20\%\cdot(X-49000)+15\%(49000-12500) = 11000 \implies 0.2\cdot X-9800+5475 = 11000 \implies 0.2\cdot X = 15325$$

Therefore, $X = \dfrac{15325}{0.2} = \$76,625$.

Exercises

1. The average price of a loaf of bread was \$1.55 last year. What is the percentage increase in the price if it costs \$2.25 this year?

2. A store is giving 15% off on all of their products. How much was the original price of an item if its price is \$153 after the discount?

3. In 2000-2001, 48% of the bachelors' degrees and 29% of masters' degrees awarded in Mathematics were earned by female students. If the number of male students who got bachelor degrees and master degrees were 4082 and 426 respectively, find the number of females who were awarded bachelor and master degrees in that year?

4. In a high school 16% of the students will be graduated this year of which 46% are girls. Find the number of girls in the high school, if we know that 108 boys are graduated this year and the number boys are $\frac{2}{3}$ of girls in the school?

5. Charles bought a new house recently. The final price after adding 16% taxes and \$1200 notary fees was \$517, 400. Find the listing price.

6. Sarah got a new job in a furniture store with a biweekly fixed payment of \$500. On top of that she receives 5% commissions on the items she sells.

   (a) If she assists the customers to buy \$10, 000 worth of furnitures on average every two weeks, how much is her biweekly pay stub before any taxes?

   (b) How much does she need to sell weekly if she wants to have at least \$1200 biweekly payment before taxes?

7. The federal income tax follows the table:

annual incomes	percentage
\$0 – \$12, 500	0%
\$12, 501 – \$49, 000	15%
\$49, 001 – \$98, 000	20%
\$98, 001 – \$152, 00	26%

   (a) Calculate your federal tax if you have \$113, 000 annual income.

   (b) Find John's salary knowing that he pays almost \$17000 for his federal tax.

6.2 Ratios

A **ratio** is the indicated quotient of two mathematical values, in other words, the relationship in quantity, amount, or size between two or more things. If there are 16 girls and 18 boys in our class, then the ratio of the number of girls to boys is $\frac{16}{18} = \frac{8}{9}$. This ratio is also denoted by $16 : 18$ or equivalently $8 : 9$.

So, a ratio is the same as a quotient/fraction $\frac{a}{b}$ which is presented by $a : b$. It is easily seen that a percentage is a special case of ratio. If 84% of the students who had Math-200 passed the course in the last semester, then the ratio of the students who passed the course to the whole number of students who had this course during the last semester was $84 : 100$ or equivalently $21 : 25$. This is ratio to the whole, however you may say that the ratio of the students who passed the course to those who failed the course was $84 : 16$ or $21 : 4$.

We can also have a ratio between more than two quantities, for instance, the ratio of cups of sugar to flour to cocoa powder in a cake recipe is $2 : 3 : 1$. This means that for any two cups of sugar, we need three cups of flour and a cup of cocoa powder.

In the ratio problems, we often need to apply a **proportion** that is an equality between two ratios. We use **cross multiplication** to solve a proportion for a missing term.

$$a : b = c : d \quad \Longleftrightarrow \quad \frac{a}{b} = \frac{c}{d} \quad \Longleftrightarrow \quad a \cdot d = b \cdot c$$

Examples

(1) A club has 21 members of which 13 are male. What is the ratio of females to all club members? With the same ratio, how many males are among 147 new members?

Solution. It is easy to see that the ratio of the women to the all members is $8 : 21$ and to answer the second part, let us assume that m is the number of males in 147 new members, then:

$$\frac{13}{21} = \frac{m}{147} \quad \Longleftrightarrow \quad 21 \cdot m = 13 \cdot 147 \quad \Longleftrightarrow \quad m = 91.$$

(2) In a final chess match, a reward of \$2250 needs to be shared in a ratio of $2 : 7$ between the second and the first place. How many dollars, will the second place receive?

Solution. The ratio of the second place to the whole is $2 : 9$ which yields to the following

proportion:

$$\frac{2}{9} = \frac{x}{2250} \implies 9 \cdot x = 2 \cdot 2250 = 4500 \iff x = \$500$$

(3) The ratio of the first year students to the rest of the students in a college is 2 : 9. Find the number of first year students if we know that there are 17820 girls registered in the college and 55% of the students are boys.

Solution. Let us find A the number of all students. Since the girls are 45% of the number of students, then

$$45\%A = 0.45 \cdot A = 17820 \implies A = \frac{17820}{0.45} = 39600,$$

Using the given ratio, we have the ratio of 2 : 11 between the first year students and all of the students. Therefore

$$\frac{2}{11} = \frac{x}{39600} \implies x = \frac{2 \cdot 39600}{11} = 7200$$

So, there are 7200 new students in the college.

6.3 Mixing Problems

Mixture problems or mixing problems are problems where items **or** different substances are mixed together and we have to determine some quantity (percentage, price,...) of the resulting mixture or the initial quantities instead.

In the following examples you will see some mixture problems. Note that in these problems, the weight or the volume of the mixture is the sum of the weights or the volumes in the substances that are mixed whereas the percentage or the price will not be the sum of percentages or prices of the initial items in the mixture.

Examples

(1) You add 100 ml of a 25% alcohol solution to a 200 ml of a 10% alcohol solution to obtain another solution. Find the amount of alcohol in the final solution.

Solution. We denote the percentage of the mixture as $x\%$ and as we see above the volume of the mixture will be 300 ml. So,

$$0.25 \cdot 100 + 0.10 \cdot 200 = x \cdot 300 \implies 45 = 300x \implies x = \frac{45}{300} = 0.15$$

Therefore the solution is a 15% alcohol solution.

(2) A farmer mixes some peanuts that sell for \$2.50/kg with almonds that sell for \$5/kg to make a 12-kg mixture worth \$3/kg. How many kilograms of peanuts are there in the mixture?

Solution. In this example, we set x kilograms for the weight of peanuts in the mixture, and therefore the weight of almonds will be $12 - x$ kg. Now set up the following equation and find x.

$$2.50 \cdot x + 5 \cdot (12 - x) = 3 \cdot (12) = 36 \implies 2.5x + 60 - 5x = 36 \implies 24 = 2.5x \implies x = 9.6$$

(3) How many litres of pure water must be added to 1000 litres of a 3% saline solution to make it a 1% saline solution?

Solution. Let us have x litres for the volume of the pure water which is in fact 0% saline solution. Note that the volume of the mixture is $1000 + x$ litres in the following equation:

$$0 \cdot x + 0.03 \cdot 1000 = 0.01 \cdot (1000 + x) \implies 30 = 10 + 0.01x \implies x = 2000 \text{ litres}$$

(4) How many litres of 20% alcohol solution should be added to 40 litres of a 50% alcohol solution to make a 30% solution?

Solution. In the following equation, x denotes the volume of the 20% alcohol solution:

$$0.20 \cdot x + 0.50 \cdot 40 = 0.30 \cdot (40 + x) \implies 0.2x + 20 = 12 + 0.3x \implies 8 = 0.1x \implies x = 80 \text{ litres}$$

(5) You need a 15% acid solution for a certain test, but your supplier only ships a 10% solution and a 30% solution. Rather than pay the hefty surcharge to have the supplier make a 15% solution, you decide to mix 10% solution with 30% solution, to make your own 15% solution. You need 10 litres of the 15% acid solution. How many litres of 10% solution and 30% solution should you use?

Solution. Let us use x litres for 30% solution and consequently, the volume of the 10% in the mixture will be $10 - x$ litres. So:

$$0.30 \cdot x + 0.10 \cdot (10 - x) = 0.15 \cdot 10 \implies 0.3x + 1 - 0.1x = 1.5 \implies x = 2.5 \text{ litres}$$

(6) Sterling Silver is 92.5% pure silver. How many grams of Sterling Silver must be mixed to a 90% Silver alloy to obtain a 500g of a 91% Silver alloy?

Solution. x is the weight of Sterling Silver in the equation.

$$0.925 \cdot x + 0.90 \cdot (500 - x) = 0.91 \cdot 500 \implies 0.925x + 450 - 0.9x = 455 \implies 0.025x = 5 \implies x = 200\ g$$

Exercises

1. If you mix 150 ml of a 25% alcohol solution to a 250 ml of a 10% alcohol solution, what will be the amount of alcohol in the final solution?

2. Coffee worth $1.05 per pound is mixed with coffee worth 85¢ per pound to obtain 20 pounds of a mixture worth 90¢ per pound. How many pounds of each type are used?

3. How many grams of pure water must be added to 50 g of pure acid to make a solution that is 40% acid?

4. A chemist has 600 ml of a solution of 20% alcohol on hand, and she wants to mix it with enough 50% alcohol solution to turn it into a 30% alcohol solution. How much of the 50% solution will she need?

5. Solution A is 50% hydrochloric acid, while solution B is 75% hydrochloric acid. How many litres of each solution should be used to make 100 litres of a solution which is 60% hydrochloric acid?

6. A farmer has two types of cream, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up with 90 gallons of 20% butterfat?

7. How many kilograms of tea that costs $4.20 per kg must be mixed with 12 kg of tea that costs $2.25 per kg to make a mixture that costs $3.40 per kg?

8. How many litres of Pure water is to be added to L litres of a 5% saline solution to make it a 2% saline solution? (Find the final answer in terms of L.)

6.4 Final Answers to Exercises

Section 6.1

1. 45.16%
2. \$180
3. Bachelor degree: 3768 girls and Master degree 174 females
4. 750 girls (and 500 boys)
5. $\$445,000$
6. **(a)** \$1000, **(b)** \$7000
7. **(a)** $\$19,175$, **(b)** $\$104,635$

Section 6.3

1. 15.625%
2. 5 pounds of \$1.05 worth coffee and 15 pounds of 85¢.
3. 75 g
4. 300 ml
5. 60 L of A and 40 L of B.
6. 30 gallons of 24% and 60 gallons of the other one.
7. 17.25 kg
8. $\dfrac{3}{2}L$

Chapter 7

Factoring and Quadratic Equations

7.1 Factoring

From the early years of schooling we all get to know what factorization of numbers mean. For instance, the number 600 can be factored (i.e., expressed) as

$$600 = 2 \times 2 \times 2 \times 3 \times 5 \times 5 = 2^3 \cdot 3 \cdot 5^2.$$

In the above factorization, the numbers $2, 3$ and 5 are primes[1], and are said to be the **prime factors** or the **irreducible factors** of 600. In a manner similar to this, we shall now seek for factorization of polynomials. In other words, given a polynomial, we would like to express it as the product of its irreducible factors. This problem might not have a unique answer. More specifically, the notion of irreducibility heavily depends upon "what sort of coefficients are allowed". To elaborate on this, consider the following polynomial:

$$P(x) = x^2 - 3.$$

If one does not allow irrational coefficients, then $P(x)$ cannot be factored, a fact which will be explained in the sequel. However, if one allows irrational numbers, one then immediately

[1] A prime number, by definition, is a positive integer which is divisible by 1 and itself **only**.

gets the following factorization:

$$P(x) = (x + \sqrt{3})(x - \sqrt{3}).$$

Convention. *In this section we are concerned exclusively with factoring polynomials over the integers.* This means specifically that all the coefficients of all the polynomials involved must be among the integer numbers:

$$... - 3, -2, -1, 0, 1, 2, 3, ...$$

As a consequence, we shall not consider $x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$ as a factorization. Nor do we consider

$$2x + 5 = 2(x + \frac{5}{2})$$

a factorization of the polynomial $2x + 5$, even though this is a valid equality.

Before we proceed further, we should perhaps also remark that the rules of factoring polynomials are closely related to the multiplication rules which we extensively explained in an earlier section. In fact, every multiplication done in the section **Polynomials I** can be seen also as a factorization; the point is that the left hand side of any of the multiplications carried out in that section is a factorization of the right hand side of the resulting equality. In a certain sense, we are playing the same game, but now the other way round!

Techniques of Factorization:

In what follows, we introduce a number of rather simple yet powerful techniques for factoring polynomials.

(I) Take out the Greatest Common Factor (G.C.F): The idea here is to pull out as much as we can, based on the formal equality

$$AK + AL + \cdots = A(K + L + \cdots).$$

Examples

(1)

$$3x^2 - 6x + 12 = 3(x^2 - 2x + 4),$$

note that 3 is the only common factor (i.e., the G.C.F) which we can pull out;

(2)

$$8x^3 - 12x^2 + 24x = 4x(2x^2 - 3x + 6),$$

as the G.C.F $= 4x$;

(3)

$$2x^3y - 6x^2y^2 + 14xy^3 = 2xy(x^2 - 3xy + 7y^2),$$

as the G.C.F $= 2xy$;

(4)

$$2x^2(x+1) - 7(x+1) = (x+1)(2x^2 - 7),$$

as the G.C.F $= (x+1)$;

(5)

$$4x^2(x-2) - x(x-2)^2 = x(x-2)(4x - (x-2)) = x(x-2)(3x+2),$$

as the G.C.F $= x(x-2)$.

(II) Factoring by Grouping: Sometimes one puts the terms of a given expression into groups and in each group one takes out the G.C.F. of that group; this would often open a window towards the factorization! Let us illustrate this technique through a few examples:

Examples

(6)

$$\begin{aligned} x^2 + xy + 3x + 3y &= \underbrace{x^2 + xy}_{\text{group 1}} + \underbrace{3x + 3y}_{\text{group 2}} \\ &= x(x+y) + 3(x+y) \qquad \text{cf. Example } (\mathbf{4}) \\ &= (x+y)(x+3); \end{aligned}$$

(7)

$$\begin{aligned} 4x^2 + 10x - 6x - 15 &= (\underbrace{4x^2 + 10x}_{\text{group 1}}) - (\underbrace{6x + 15}_{\text{group 2}}) \\ &= 2x(2x+5) - 3(2x+5) \\ &= (2x+5)(2x-3); \end{aligned}$$

And here is a different grouping used to factor the same expression:

$$\begin{aligned} 4x^2+10x-6x-15 &= (\underbrace{4x^2-6x}_{\text{group 1}})+(\underbrace{10x-15}_{\text{group 2}}) \\ &= 2x(2x-3)+5(2x-3) \\ &= (2x-3)(2x+5). \end{aligned}$$

(III) Factoring Quadratic Expressions ax^2+bx+c:

Factoring Special Trinomials of the Form x^2+bx+c

A quadratic expression of the type x^2+bx+c is factorable over integers if there exist two *integer numbers* "m" *and* "n" such that

$$m+n=b \quad \text{and} \quad mn=c.$$

If that is the case, then we have the factorization:

$$x^2+bx+c=(x+m)(x+n),$$

which can be easily verified as follows

$$(x+m)(x+n)=x^2+mx+nx+mn=x^2+(m+n)x+mn=x^2+bx+c.$$

Examples

(8) $x^2+6x+5=(x+5)(x+1)$ since $5+1=6$ and $(5)\cdot(1)=5$.

(9) $x^2+7x-30=(x-3)(x+10)$ since $-3+10=7$ and $(-3)\cdot(10)=-30$.

Factoring General Trinomials of the Form ax^2+bx+c

Given the more general quadratic expression ax^2+bx+c, we look for two numbers "m" and "n" whose sum is b, and whose product is, not just c, but ac:

$$m+n=b \quad \text{and} \quad mn=ac.$$

It is with the help of these two numbers that we form the right groups and then proceed, as illustrated above, to factor the given expression. The comment we would like to make here

is that this is possible (i.e., the expression $ax^2 + bx + c$ is factorable over the integers) if and only if[2] $b^2 - 4ac$ is a *perfect square*, that is to say, if and only if

$$b^2 - 4ac = 0, 1, 4, 9, 16....$$

(10) To factor $5x^2 - 7x - 6$, we look for two integers whose sum is -7 and whose product is $5 \cdot (-6) = -30$. As the numbers are -10 and 3, we proceed as follows:

$$\begin{aligned} 5x^2 - 7x - 6 &= 5x^2 - 10x + 3x - 6 \\ &= 5x(x-2) + 3(x-2) \\ &= (x-2)(5x+3). \end{aligned}$$

(11) To factor $10x^2 - 23x + 12$, we look for two integers whose sum is -23 and whose product is $10 \cdot 12 = 120$. As the numbers are -15 and -8, we proceed as follows:

$$\begin{aligned} 10x^2 - 23x + 12 &= 10x^2 - 15x - 8x + 12 \\ &= 5x(2x-3) - 4(2x-3) \\ &= (2x-3)(5x-4). \end{aligned}$$

(IV) Using Special Identities: If possible, we apply any one of the following identities:

- $a^2 - b^2 = (a-b)(a+b)$;
- $a^2 + 2ab + b^2 = (a+b)^2$ and $a^2 - 2ab + b^2 = (a-b)^2$;
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ and $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

Note that, in applications, a and/or b may be replaced by anything: number(s), variable(s), etc.

Examples

(12)

$$\begin{aligned} x^2 + 10x + 25 &= x^2 + 2 \cdot x \cdot 5 + 5^2 \\ &= (x+5)^2; \qquad [\text{using } \ a^2 + 2ab + b^2 = (a+b)^2] \end{aligned}$$

[2] We will learn more about this criterion in later sections.

(13)

$$\begin{aligned} y^4 - 6y^2 + 9 &= (y^2)^2 - 2 \cdot y^2 \cdot 3 + 3^2 \\ &= (y^2 - 3)^2; \qquad [\text{using } \; a^2 - 2ab + b^2 = (a-b)^2] \end{aligned}$$

(14)

$$\begin{aligned} X^6 t^3 - 8Y^3 &= (X^2 t)^3 - (2Y)^3 \\ &= (X^2 t - 2Y)((X^2 t)^2 + (X^2 t)(2Y) + (2Y)^2) \\ &= (X^2 t - 2Y)(X^4 t^2 + 2X^2 Y t + 4Y^2); \end{aligned}$$

(15)

$$\begin{aligned} (x^2 + y)^2 - 25 &= (x^2 + y)^2 - 5^2 \\ &= (x^2 + y - 5)(x^2 + y + 5). \end{aligned}$$

(V) Using all the Techniques Discussed so far(!): Instead of further talking, let us simply apply simultaneously all the methods discussed above to factor more complicated expressions:

Examples

(16)

$$\begin{aligned} 36x^4 - 25x^2 + 4 &= 36x^4 - 9x^2 - 16x^2 + 4 \\ &= 9x^2(4x^2 - 1) - 4(4x^2 - 1) \\ &= (4x^2 - 1)(9x^2 - 4) \\ &= (2x - 1)(2x + 1)(3x - 2)(3x + 2); \end{aligned}$$

(17)

$$\begin{aligned} (x^2 - 9)^2 + 8x(x^2 - 9) &= (x^2 - 9)[(x^2 - 9) + 8x] \\ &= (x - 3)(x + 3)(x^2 + 8x - 9) \\ &= (x - 3)(x + 3)(x - 1)(x + 9); \end{aligned}$$

(18)

$$\begin{aligned} x^6 - 64 &= (x^3 - 8)(x^3 + 8) \\ &= (x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4). \end{aligned}$$

Note that, according to the criterion stated at the end of **<u>Case III</u>**, the last two quadratic brackets are irreducible as neither $2^2 - 4 \cdot 1 \cdot 4$ nor $(-2)^2 - 4 \cdot 1 \cdot 4$ is a perfect square!.

Exercises

1. Use the first two techniques, "G.C.F." and "Factoring by Grouping", to factor the following expressions completely.

   (a) $5xy - 15x^2$

   (b) $18a^2b^3 - 6ab^2$

   (c) $49x^2 - 14xy^2 + 28yx$

   (d) $4a^3b^2 + 10a^2b^3 - 6a^4b^2$

   (e) $12x^3y - 20x^2y^2 + 8xy$

   (f) $20a^4 - 15a^2b^3 + 10b^4$

   (g) $18x^5y^2 - 12x^3y^4 + 24x^2y^3$

   (h) $5x(x-1) + 4(x-1)$

   (i) $2x(x+5) - 3(x+5)$

   (j) $x^2(2x-1) + (2x-1)$

   (k) $7y^2(y+1) + 14y(y+1)$

   (l) $24a(b+1)^2 - 8(b+1)^2$

   (m) $x^2 + 3x + 2xy + 6y$

   (n) $x^2 + 7x - 2x - 14$

   (o) $4y^2 + 10y - 6y - 15$

   (p) $3ab - b^2 + 3a - b$

   (q) $3x^3 + 3x^2 - 2x - 2$

   (r) $x - 1 - xy + y$

   (s) $10x^4 - 8x^3 - 5x^2 + 4x$

2. Factor each trinomial (quadratic form).

   (a) $x^2 + 4x + 3$

   (b) $t^2 + t - 20$

   (c) $y^2 + 2y - 8$

   (d) $x^2 - 13x + 42$

   (e) $a^2 - 2a - 63$

   (f) $x^2 - x - 56$

   (g) $y^2 - 9y + 20$

   (h) $x^2 + 12x + 35$

   (i) $x^2 - 5yx + 6y^2$

   (j) $x^2 + 2yx - 3y^2$

   (k) $x^2 - 2yx - 24y^2$

   (l) $3x^2 + 8x + 5$

   (m) $2y^2 + 5y - 3$

   (n) $4x^2 - 12x + 5$

   (o) $3a^2 + 10a - 8$

   (p) $2t^2 - 6t - 20$

(q) $-x^2 + 6x + 16$

(r) $10x^2 - 23x + 12$

(s) $7y^2 - 27y - 4$

(t) $-2x^2 + 13x - 15$

(u) $-3x^2 + 22x - 7$

(v) $-6x^2 + 17x - 5$

3. Use appropriate identities to factor each polynomial.

(a) $x^2 - 36$

(b) $4x^2 - 1$

(c) $25x^2 - 49$

(d) $1 - 64y^2$

(e) $16x^2 - 121y^2$

(f) $4y^2 - 25$

(g) $x^4 - 9y^2$

(h) $4a^2b^2 - 1$

(i) $x^2 - 10x + 25$

(j) $t^2 + 22t + 121$

(k) $x^2 - 6x + 9$

(l) $4x^2 + 20x + 25$

(m) $36x^2 - 12x + 1$

(n) $1 - 4x + 4x^2$

(o) $16y^2 - 56y + 49$

(p) $9x^2 - 24x + 16$

(q) $x^3 - 8$

(r) $x^3 + 27$

(s) $8x^3 - 125$

(t) $27t^3 - 64$

(u) $8x^3 + 27y^3$

(v) $125a^3 - 64b^3$

4. Factor Completely.

(a) $128x^4 - 8y^2x^2$

(b) $10x^4 - 270x$

(c) $x^4 - x^2$

(d) $x^3 + 3x^2 - x - 3$

(e) $5(x+1) - 6(x+1)^2$

(f) $y^2x - 3y^2 - 4x + 12$

(g) $t^2(t-2) - (t-2)$

(h) $24x^4 - 16x^3 - 81x + 54$

(i) $18x^2y - 8x^4y$

(j) $(x^2 - 9)(x^2 - x - 2)$

(k) $a^3(a+b)^2 + b^3(a+b)^2$

(l) $9 - (2x+1)^2$

(m) $(5x+7)^2-16$

(n) x^4-x^2-20

(o) x^4-16

(p) $12x^3y-30x^2y-18xy$

(q) $8x^4(x-4)-27x(x-4)$

(r) $(x+1)^2-(x+1)-6$

(s) $7x^4+7x^3-140x^2$

(t) x^6-64

7.2 Solving Quadratic Equations

Definition 7.1 *An equation in x that can be (re)written in the "standard" form of*

$$ax^2 + bx + c = 0,$$

where a, b and c are real numbers and $a \neq 0$, is called a **quadratic equation**.

Given a quadratic equation $ax^2 + bx + c = 0$, we call the quantity

$$\Delta = b^2 - 4ac$$

the **discriminant** of the equation. One significance of Δ comes from the following criterion concerning the number of solutions:

(I) If $\Delta > 0$, then the equation has **two distinct** solutions;

(II) If $\Delta = 0$, then the equation has **one repeated** solution, that is to say, the two solutions are equal;

(III) And finally if $\Delta < 0$, then **no real number** is a solution to the equation!

Examples

(1) The equation $5x^2 - 3x - 2 = 0$ has two solutions, as $\Delta = (-3)^2 - 4 \cdot 5 \cdot (-2) = 49 > 0$; [the solutions are $x_1 = 1$ and $x_2 = -2/5$].

(2) The equation $4x^2 + 12x + 9 = 0$ has only one solution, since $\Delta = 12^2 - 4 \cdot 4 \cdot 9 = 0$; [the solutions are $x_1 = x_2 = -3/2$].

(3) And the equation $x^2 + x + 1 = 0$ has no solutions at all, as $\Delta = 1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$.

The other significance of the discriminant is that it participates in **the quadratic formula**:

Quadratic Formula:

Given the quadratic equation $ax^2 + bx + c = 0$ with $\Delta \geq 0$, the solution(s) to the equation are given by

$$\boxed{x_1,\ x_2 = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Remark 1. If $\Delta = 0$, then the formula above will reduce to a *unique solution:* $x_1 = x_2 = \dfrac{-b}{2a}$.

Examples

(4) Solve for x: $5x^2 - 3x - 2 = 0.$

Solution. As $\Delta = 49$, we have

$$x_1,\ x_2 = \frac{-(-3) \pm \sqrt{49}}{2 \cdot 5} = \frac{3 \pm 7}{10} = 1,\ \ -\frac{2}{5}.$$

(5) Solve for x: $4x^2 + 12x + 9 = 0.$

Solution. As $\Delta = 0$, we get

$$x_1 = x_2 = -\frac{12}{2 \cdot 4} = -\frac{3}{2}.$$

(6) Solve for x: $x^2 = 4x - 1.$

Solution. The standard form of the equation is $x^2 - 4x + 1 = 0$, and thus $\Delta = 12$. Therefore, we have

$$x_1,\ x_2 = \frac{-(-4) \pm \sqrt{12}}{2 \cdot 1} = \frac{4 \pm 2\sqrt{3}}{2} = \frac{2(2 \pm \sqrt{3})}{2} = 2 \pm \sqrt{3}.$$

(7) Solve for x: $x^2 = x - 1.$

Solution. The standard form of the equation is $x^2 - x + 1 = 0$; as $\Delta = -3 < 0$, the equation has no solution.

Remark 2. Sometimes, instead of applying the quadratic formula displayed in the box above, one may use the so-called **Square-Root Property**

$$\boxed{\begin{array}{c} \text{If} \quad X^2 = k, \quad (k > 0) \\[2ex] \text{then} \quad X = \pm\sqrt{k} \end{array}}$$

Examples

(8) Solve for x: $\qquad 2(x-1)^2 - 1 = 5.$

Solution. First we simplify and then we use Square Root Property.:

$$\begin{aligned} 2(x-1)^2 = 6 \quad &\Longrightarrow \quad (x-1)^2 = 3 \\ &\Longrightarrow \quad x - 1 = \pm\sqrt{3} \\ &\Longrightarrow \quad x = 1 \pm \sqrt{3}. \end{aligned}$$

(9) Solve for x: $\qquad (2x+1)^2 = 5.$

Solution. Using the Square Root Property, we may write

$$\begin{aligned} (2x+1)^2 = 5 \quad &\Longrightarrow \quad 2x + 1 = \pm\sqrt{5} \\ &\Longrightarrow \quad 2x = -1 \pm \sqrt{5} \\ &\Longrightarrow \quad x_1, x_2 = \frac{-1 \pm \sqrt{5}}{2}. \end{aligned}$$

(10) Solve for a: $\qquad a^2 + 6a = 3.$

Solution. First we convert the left side into a **perfect square** and then we apply Square Root Property:

$$\begin{aligned} a^2 + 6a + 9 = 3 + 9 \quad &\Longrightarrow \quad (a+3)^2 = 12 \\ &\Longrightarrow \quad a + 3 = \pm\sqrt{12} \\ &\Longrightarrow \quad a = -3 \pm 2\sqrt{3}. \end{aligned}$$

Solving Equations by Factoring

Given a standard quadratic equation $ax^2+bx+c=0$, where the expression ax^2+bx+c can be factorized, the equation can be easily solved using **the Zero-Factor Property** which states that:

$$\boxed{\text{If} \quad AB=0 \quad \text{then} \quad A=0 \ \text{ or } B=0.}$$

First we make sure that the equation is written in standard form, then we factor the expression completely and finally we use the Zero-Factor Property to find the solutions. This method can be applied to any polynomial equation of any degree that is factorable too.

Examples

(11) Solve for x: $\quad x^2-2x-3=0.$

Solution.

$$\begin{aligned} x^2-2x-3=0 &\implies (x-3)(x+1)=0 \\ &\implies x-3=0, \quad x+1=0 \\ &\implies x=3, \quad x=-1. \end{aligned}$$

(12) Solve for x: $\quad 3x^2=5x-2.$

Solution.

$$\begin{aligned} 3x^2-5x+2=0 &\implies (3x-2)(x-1)=0 \\ &\implies 3x-2=0, \quad x-1=0 \\ &\implies x=\frac{2}{3}, \quad x=1. \end{aligned}$$

(13) Solve for x: $\quad 2x^3-x^2=8x-4.$

Solution.

$$\begin{aligned} 2x^3-x^2-8x+4=0 &\implies (x^2-4)(2x-1)=0 \\ &\implies (x-2)(x+2)(2x-1)=0 \\ &\implies x=2, \quad x=-2, \quad x=\frac{1}{2}. \end{aligned}$$

Exercises

1. Solve the equations.

   (a) $x^2 - 2x - 2 = 0$

   (b) $x^2 + 3x = 3$

   (c) $x^2 - 18 = 0$

   (d) $x^2 = 4x + 3$

   (e) $a^2 - 10a + 22 = 0$

   (f) $2x^2 + 1 = 5x$

   (g) $3y^2 - 4y - 2 = 0$

   (h) $4x^2 + 4x - 7 = 0$

   (i) $2a - a^2 = -12$

   (j) $9x + 5 = 2x^2$

   (k) $5a^2 - 4a + 3 = 0$

   (l) $3x^2 = 6x - 2$

   (m) $x^2 = 10x + 5$

   (n) $(x - 1)^2 = 6$

   (o) $2(x + 5)^2 = 16$

   (p) $(y - 5)^2 = 2y$

   (q) $4x^2 + 4x = 7$

   (r) $(x + 3)(x - 2) = 2$

   (s) $(x + 2)(2x - 1) = x^2 - 1$

   (t) $(x^2 - 1)(x + 2) = x^3 - 4$

2. Solve the equations by factoring.

   (a) $x^2 - 4x + 3 = 0$

   (b) $x^2 - 3x - 10 = 0$

   (c) $a^2 = a + 30$

   (d) $9x^2 - 3x = 2$

   (e) $x + 6 = 2x^2$

   (f) $3y^2 + 22y - 16 = 0$

   (g) $3x^2 + 12x = 0$

   (h) $10y^2 = 5y$

   (i) $x^2 = 3x$

   (j) $2x^2 = 8x$

   (k) $x^2 + 8x + 16 = 0$

   (l) $a^2 + 81 = 18a$

   (m) $4x^2 + 20x + 25 = 0$

   (n) $x^2 = 36$

   (o) $y^2 - 121 = 0$

   (p) $4x^2 - 49 = 0$

(q) $5x^2 = 7x + 6$

(r) $x^3 - 10x^2 - 39x = 0$

(s) $8n^3 + 25n = 30n^2$

(t) $81x^3 + 100x = 180x^2$

(u) $(x + 3)^2 - 4 = 0$

3. Use Square Root Property to solve the following equations.

(a) $2x^2 - 16 = 0$

(b) $3x^2 + 17 = 0$

(c) $(2x)^2 = 24$

(d) $4(x + 3)^2 - 3 = 17$

(e) $(a - 4)^2 = 8$

(f) $(2x - 5)^2 - 180 = 0$

(g) $4(t - 2)^2 - 25 = 0$

(h) $(x - \frac{1}{3})^2 = \frac{5}{9}$

(i) $(\frac{1}{2}y + 3)^2 = 12$

(j) $x^2 + 14x + 49 = 18$

(k) $x^2 - 10x = 7$

4. Solve the equations.

(a) $3(3x - 1)^2 - 8 = 19$

(b) $(x + 2)^2 - 5(x + 3) + 6 = 0$

(c) $x^5 = 10x^3 - 9x$

(d) $a^2 + \frac{a}{2} = \frac{1}{8}$

(e) $\sqrt{2}x^2 - 3x + \sqrt{2} = 0$

(f) $9(2x + 6)^2 - 12 = 0$

(g) $y^2 + (y + 1)^2 = 41$

(h) $121x = 144x^3$

(i) $t(t + 8) = -15$

(j) $x^3 - 6x^2 + 7x = 0$

(k) $x^4 - 12x^2 + 32 = 0$

(l) $y^2 - 3\sqrt{2}y = 3$

(m) $(4x - 1)(2x - 3) = -2$

(n) $\dfrac{x + 1}{2x - 1} = \dfrac{3x}{x - 1}$

(o) $\dfrac{-3}{2x + 1} = \dfrac{x - 1}{x^2}$

5. If 2 is one of the solutions of the equation $x^2 - 8x + k = 0$, find k and the other solution.

6. If -3 is one of the solutions of the equation $x^2 - kx + 3 = 0$, find k and the other solution.

7. Find k such that $-1+\sqrt{5}$ is a solution of $x^2+2x+k=0$.

8. Find the other solution of $x^2-2x-k=0$ if $1+\sqrt{2}$ is one of its solutions.

9. The sum of the squares of two consecutive integers is 313. Find the integers.

10. The sum of the squares of two consecutive even integers is 100. Find them.

11. The sum of the squares of two consecutive odd integers is 290. Find them.

12. The sum of the squares of three consecutive integers is 149. Find them.

13. Find two integers whose product is -12 and whose sum is -11.

14. Find two numbers whose sum is 2 and whose product is -2.

15. The product of two consecutive integers is 11 more than their sum. Find them.

16. The sum of a number and its square is $\frac{10}{9}$. Find the number(s).

17. The sum of double a number and its square is 1. Find the number(s).

18. The sum of an integer and its square is 7 times the next consecutive integer. Find them.

19. The difference of two numbers is 3 and their product is 270. Find them.

20. A man is 5 times as old as his son and the sum of the squares of their ages is 2106. How old is the father?

21. A girl is 5 years younger than her sister and the product of their ages is 204. How old is the older sister?

22. Find the dimensions of a rectangle whose area is 21 m^2 and its length is 1 more than double its width.

23. Find the perimeter of a rectangle whose area is 80 in^2 and its width is 11 inch less than its length.

24. The width of a rectangle is 5 feet less than half of its length. Find the dimensions of the rectangle if its area is 48 ft^2

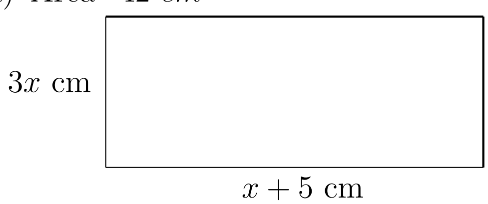
25. The height of a triangle is 5 cm more than double its base. Find the base if the area of the triangle is 450 cm^2.

26. The base of a triangle is 12 m less than 3 times its height. Find the base and the height if the area of the triangle is 96 m^2.

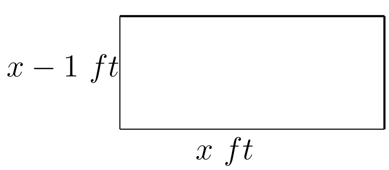
27. The sum of the base and the height of a triangle is 26 ft. Find them if the area of the triangle is 51 ft^2.

28. Find x in each of the following geometric objects.

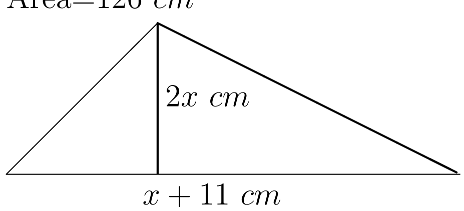
   (a) Area=42 cm^2



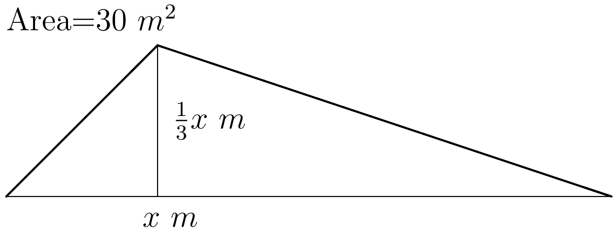
   (b) Area=5 ft^2



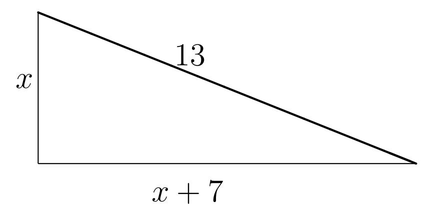
   (c) Area=126 cm^2



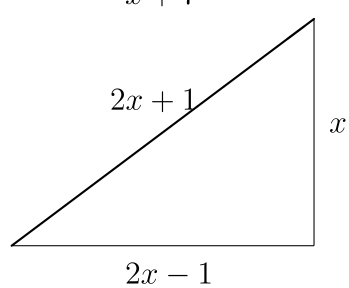
   (d) Area=30 m^2



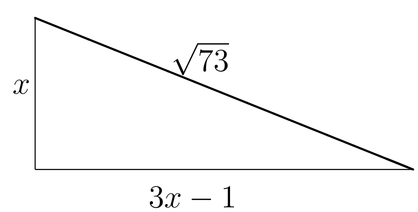
   (e)



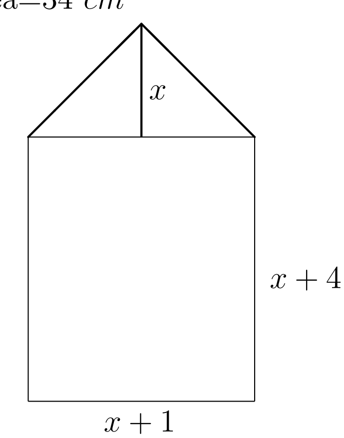
   (f)



   (g)



   (h) Area=34 cm^2



7.3 Final Answers to Exercises

Section 7.1

1. (a) $5x(y-3x)$
   (b) $6ab^2(3ab-1)$
   (c) $7x(7x-2y^2+4y)$
   (d) $2a^2b^2(2a+5b-3a^2)$
   (e) $4xy(3x^2-5xy+2)$
   (f) $5(4a^4-3a^2b^3+2b^4)$
   (g) $6x^2y^2(3x^3-2xy^2+4y)$
   (h) $(x-1)(5x+4)$
   (i) $(x+5)(2x-3)$
   (j) $(2x-1)(x^2+1)$
   (k) $7y(y+1)(y+2)$
   (l) $8(b+1)^2(3a-1)$
   (m) $(x+3)(x+2y)$
   (n) $(x+7)(x-2)$
   (o) $(2y+5)(2y-3)$
   (p) $(3a-b)(b+1)$
   (q) $(3x^2-2)(x+1)$
   (r) $(x-1)(1-y)$
   (s) $x(5x-4)(2x^2-1)$

2. (a) $(x+3)(x+1)$
   (b) $(t+5)(t-4)$
   (c) $(y+4)(y-2)$
   (d) $(x-7)(x-6)$
   (e) $(a-9)(a+7)$
   (f) $(x-8)(x+7)$
   (g) $(y-4)(y-5)$
   (h) $(x+5)(x+7)$
   (i) $(x-2y)(x-3y)$
   (j) $(x+3y)(x-y)$
   (k) $(x-6y)(x+4y)$
   (l) $(3x+5)(x+1)$
   (m) $(2y-1)(y+3)$
   (n) $(2x-5)(2x-1)$
   (o) $(3a-2)(a+4)$
   (p) $2(t+2)(t-5)$
   (q) $(-x-2)(x-8)=-(x+2)(x-8)$
   (r) $(2x-3)(5x-4)$
   (s) $(7y+1)(y-4)$
   (t) $(-2x+3)(x-5)$
   (u) $(3x-1)(-x+7)$
   (v) $(-2x+5)(3x-1)$

3. (a) $(x-6)(x+6)$
   (b) $(2x-1)(2x+1)$
   (c) $(5x-7)(5x+7)$
   (d) $(1-8y)(1+8y)$

(e) $(4x-11y)(4x+11y)$

(f) $(2y-5)(2y+5)$

(g) $(x^2-3y)(x^2+3y)$

(h) $(2ab-1)(2ab+1)$

(i) $(x-5)^2$

(j) $(t+11)^2$

(k) $(x-3)^2$

(l) $(2x+5)^2$

(m) $(6x-1)^2$

(n) $(1-2x)^2$

(o) $(4y-7)^2$

(p) $(3x-4)^2$

(q) $(x-2)(x^2+2x+4)$

(r) $(x+3)(x^2-3x+9)$

(s) $(2x-5)(4x^2+10x+25)$

(t) $(3t-4)(9t^2+12t+16)$

(u) $(2x+3y)(4x^2-6xy+9y^2)$

(v) $(5a-4b)(25a^2+20ab+16b^2)$

4. (a) $8x^2(4x-y)(4x+y)$

(b) $10x(x-3)(x^2+3x+9)$

(c) $x^2(x-1)(x+1)$

(d) $(x-1)(x+1)(x+3)$

(e) $(x+1)(-6x-1)=-(x+1)(6x+1)$

(f) $(x-3)(y-2)(y+2)$

(g) $(t-2)(t-1)(t+1)$

(h) $(3x-2)(2x-3)(4x^2+6x+9)$

(i) $2x^2y(3-2x)(3+2x)$

(j) $(x-3)(x+3)(x-2)(x+1)$

(k) $(a+b)^3(a^2-ab+b^2)$

(l) $4(2+x)(1-x)$

(m) $(5x+3)(5x+11)$

(n) $(x^2-5)(x^2+4)$

(o) $(x^2+4)(x-2)(x+2)$

(p) $6xy(2x+1)(x-3)$

(q) $x(x-4)(2x-3)(4x^2+6x+9)$

(r) $(x-2)(x+3)$

(s) $7x^2(x+5)(x-4)$

(t) $(x-2)(x^2+2x+4)(x+2)(x^2-2x+4)$

Section 7.2

1. (a) $x=1\pm\sqrt{3}$

(b) $x=\dfrac{-3\pm\sqrt{21}}{2}$

(c) $x=\pm3\sqrt{2}$

(d) $x=2\pm\sqrt{7}$

(e) $a=5\pm\sqrt{3}$

(f) $x=\dfrac{5\pm\sqrt{17}}{4}$

(g) $y=\dfrac{2\pm\sqrt{10}}{3}$

(h) $x=\dfrac{-1\pm2\sqrt{2}}{2}$

(i) $a=1\pm\sqrt{13}$

(j) $x = 5$ *or* $x = \dfrac{-1}{2}$

(k) No Solution.

(l) $x = \dfrac{3 \pm \sqrt{3}}{3}$

(m) $x = 5 \pm \sqrt{30}$

(n) $x = 1 \pm \sqrt{6}$

(o) $x = -5 \pm 2\sqrt{2}$

(p) $y = 6 \pm \sqrt{11}$

(q) $x = \dfrac{-1 \pm 2\sqrt{2}}{2}$

(r) $x = \dfrac{-1 \pm \sqrt{33}}{2}$

(s) $x = \dfrac{-3 \pm \sqrt{13}}{2}$

(t) No Solution.

2. (a) $x = 1$ *or* $x = 3$

(b) $x = 5$ *or* $x = -2$

(c) $a = 6$ *or* $a = -5$

(d) $x = \dfrac{2}{3}$ *or* $x = \dfrac{-1}{3}$

(e) $x = 2$ *or* $x = \dfrac{-3}{2}$

(f) $y = -8$ *or* $y = \dfrac{2}{3}$

(g) $x = 0$ *or* $x = -4$

(h) $y = 0$ *or* $y = \dfrac{1}{2}$

(i) $x = 0$ *or* $x = 3$

(j) $x = 0$ *or* $x = 4$

(k) $x = -4$

(l) $a = 9$

(m) $x = \dfrac{-5}{2}$

(n) $x = \pm 6$

(o) $y = \pm 11$

(p) $x = \pm\dfrac{7}{2}$

(q) $x = 2$ *or* $x = \dfrac{-3}{5}$

(r) $x = 0,\ x = 13$ *or* $x = -3$

(s) $n = 0,\ n = \dfrac{5}{2}$ *or* $n = \dfrac{5}{4}$

(t) $x = 0$ *or* $x = \dfrac{10}{9}$

(u) $x = -1$ *or* $x = -5$

3. (a) $x = \pm 2\sqrt{2}$

(b) No Solution.

(c) $x = \pm\sqrt{6}$

(d) $x = -3 \pm \sqrt{5}$

(e) $a = 4 \pm 2\sqrt{2}$

(f) $x = \dfrac{5 \pm 6\sqrt{5}}{2}$

(g) $t = \dfrac{9}{2}$ *or* $t = \dfrac{-1}{2}$

(h) $x = \dfrac{1 \pm \sqrt{5}}{3}$

(i) $x = -6 \pm 4\sqrt{3}$

(j) $x = -7 \pm 3\sqrt{2}$

(k) $x = 5 \pm 4\sqrt{2}$

4. (a) $x = \frac{4}{3}$ *or* $x = \frac{-2}{3}$
   (b) $x = \frac{1 \pm \sqrt{21}}{2}$
   (c) $x = 0,\ x = \pm 1$ *or* $x = \pm 3$
   (d) $a = \frac{-1 \pm \sqrt{3}}{4}$
   (e) $x = \sqrt{2}$ *or* $x = \frac{\sqrt{2}}{2}$
   (f) $x = \frac{-9 \pm \sqrt{3}}{3}$
   (g) $y = -5$ *or* $y = 4$
   (h) $x = 0$ *or* $x = \pm\frac{11}{12}$
   (i) $t = -3$ *or* $t = -5$
   (j) $x = 0$ *or* $x = 3 \pm \sqrt{2}$
   (k) $x = \pm 2$ *or* $x = \pm 2\sqrt{2}$
   (l) $y = \frac{3\sqrt{2} \pm \sqrt{30}}{2}$
   (m) $x = \frac{5}{4}$ *or* $x = \frac{1}{2}$
   (n) No Solution.
   (o) $x = \frac{1 \pm \sqrt{21}}{10}$

5. $k = 12,\ x = 6$
6. $k = -4,\ x = -1$
7. $k = -4,\ x = -1 - \sqrt{5}$
8. $k = 1,\ x = 1 - \sqrt{2}$
9. $12,\ 13,$ or $-13,\ -12$
10. $6,\ 8,$ or $-8,\ -6$
11. $11,\ 13,$ *or* $-13,\ -11$
12. $6,\ 7,\ 8$ or $-8,\ -7,\ -6$
13. $-12,\ 1$
14. $1 \pm \sqrt{3}$
15. $4,\ 5$ or $-3,\ -2$
16. $\frac{-5}{3},\ \frac{2}{3}$
17. $-1 \pm \sqrt{2}$
18. $7,\ -1$
19. $15,\ 18$ or $-18,\ -15$

20. 45

21. 17

22. $w = 3,\ l = 7$

23. $P = 42$

24. $w = 3,\ l = 16$

25. $b = 20$

26. $b = 6\sqrt{17} - 6,\ h = 2 + 2\sqrt{17}$

27. $13 + \sqrt{67},\ 13 - \sqrt{67}$

28. (a) $x = 2$
    (b) $x = \dfrac{1 + \sqrt{21}}{2} \simeq 2.79$
    (c) $x = 7$
    (d) $x = 6\sqrt{5} \simeq 13.42$
    (e) $x = 5$
    (f) $x = 8$
    (g) $x = 3$
    (h) $x = 3$

Chapter 8

Rational Expressions and More on Equations

8.1 Rational Expressions (Fractions)

Definition 8.1 *Any fraction of the form*

$$\frac{A}{B} = \frac{A(x)}{B(x)}$$

where the numerator $A(x)$ and the denominator $B(x)$ are polynomials, is called a **rational expression** *in x. Similarly, one can define a rational expression in two variables, say x and y, as any quotient of two polynomials in x and y.*

Fundamental Property A rational expression is unchanged if both its numerator and its denominator are multiplied or divided by any non-zero factor. That is to say, for $C(x) \neq 0$, we have

$$\frac{A(x)C(x)}{B(x)C(x)} = \frac{A(x)\cancel{C(x)}}{B(x)\cancel{C(x)}} = \frac{A(x)}{B(x)}.$$

In other words, we can simplify a rational expression by getting rid of any common factor

between its numerator and denominator. Here is an example:

$$\begin{aligned}\frac{6x^2(x+1)}{9x(x+1)^2} &= \frac{2\cdot 3\cdot x\cdot x\cdot (x+1)}{3\cdot 3\cdot x\cdot (x+1)\cdot (x+1)}\\ &= \frac{2\cdot \cancel{3}\cdot \cancel{x}\cdot x\cdot \cancel{(x+1)}}{3\cdot \cancel{3}\cdot \cancel{x}\cdot \cancel{(x+1)}\cdot (x+1)}\\ &= \frac{2x}{3(x+1)}.\end{aligned}$$

Arithmetic Operations on Rational Expressions

(I) Multiplication. The rule is very simple:

$$\frac{A(x)}{B(x)}\cdot\frac{C(x)}{D(x)} = \frac{A(x)C(x)}{B(x)D(x)}.$$

We remark, once again, that whenever possible we simplify our rational expression. Of course, in order to do so, we first need to factor both the numerator and the denominator to see if there is any common factor to be canceled out.

Examples

In the following, multiply and simplify:

(1)

$$\begin{aligned}\frac{x}{x-1}\cdot\frac{x^2-1}{x^2} &= \frac{x(x^2-1)}{x^2(x-1)}\\ &= \frac{x(x-1)(x+1)}{x^2(x-1)}\\ &= \frac{x+1}{x}.\end{aligned}$$

(2)

$$\begin{aligned}\frac{x^2-1}{2x-4}\cdot\frac{x^2-4}{x^2-x-2}\cdot\frac{3x-6}{x^2+x-2} &= \frac{(x-1)(x+1)}{2(x-2)}\cdot\frac{(x-2)(x+2)}{(x-2)(x+1)}\cdot\frac{3(x-2)}{(x+2)(x-1)}\\ &= \frac{3}{2}.\end{aligned}$$

(II) Division. Once again, the rule is simple:

$$\frac{A(x)}{B(x)}\div\frac{C(x)}{D(x)} = \frac{A(x)}{B(x)}\cdot\frac{D(x)}{C(x)} = \frac{A(x)D(x)}{B(x)C(x)}.$$

Note that as $a \div b = \dfrac{a}{b}$, we may rewrite the above rule also as

$$\frac{\frac{A(x)}{B(x)}}{\frac{C(x)}{D(x)}} = \frac{A(x)}{B(x)} \div \frac{C(x)}{D(x)} = \frac{A(x)D(x)}{B(x)C(x)}.$$

Examples

Divide and simplify if possible:

(3)

$$\frac{x^3 - x}{x^2 - 3x - 4} \div \frac{x - x^2}{x^2 - 16} = \frac{x(x-1)(x+1)}{(x-4)(x+1)} \cdot \frac{(x-4)(x+4)}{-x(x-1)} = -(x+4).$$

(4)

$$\frac{\frac{x+4}{x+1}}{\frac{x+4}{x^2-1}} = \frac{(x+4)(x^2-1)}{(x+1)(x+4)} = \frac{(x-1)(x+1)}{x+1} = x - 1.$$

(III) Addition/Subtraction. The rule to add and/or subtract rational expressions is similar to that of numerical fractions:

$$\frac{A(x)}{B(x)} \pm \frac{C(x)}{D(x)} = \frac{A(x)D(x) \pm B(x)C(x)}{B(x)D(x)}.$$

Remark. It should be noted that to minimize the amount of computations it is often preferable to use the so-called L.C.M (the Least Common Multiple) of $B(x)$ and $D(x)$ rather than their product $B(x)D(x)$; With $B(x)$ and $D(x)$ factored, to get the L.C.M, one uses the common factor(s) with the greater exponent(s), together with the non-common factor(s). For instance, if

$$B(x) = 15x^3(x+2)^2(x-1)^3(x^2+5) \qquad \text{and} \qquad D(x) = 9x(x+2)(x-1)^6(x-3)^3,$$

then the L.C.M of $B(x)$ and $D(x)$ is equal to

$$45x^3(x+2)^2(x-1)^6(x^2+5)(x-3)^3.$$

Note that if we use the L.C.M, denoted by say $L(x)$, we have to modify the above formula for adding/subtracting rational expressions:

$$\frac{A(x)}{B(x)} \pm \frac{C(x)}{D(x)} = \frac{A(x)\frac{L(x)}{B(x)} \pm C(x)\frac{L(x)}{D(x)}}{L(x)}.$$

If $B(x)$ and $D(x)$ happen to have no factor in common, then this formula will reduce to the one given above.

Examples

Add/Subtrac and simplify:

(5)

$$\frac{x-2}{x+3}+\frac{x}{x-1}=\frac{(x-2)(x-1)+x(x+3)}{(x+3)(x-1)}=\frac{2x^2+2}{(x+3)(x-1)}.$$

(6)

$$\begin{aligned}\frac{x}{x-1}-\frac{2}{x^2-1} &= \frac{x(x+1)-2}{(x-1)(x+1)} \qquad \Big[\text{note that L.C.M} = (x-1)(x+1)\Big]\\ &= \frac{(x+2)(x-1)}{(x-1)(x+1)}\\ &= \frac{x+2}{x+1}.\end{aligned}$$

(7)

$$\begin{aligned}\frac{\dfrac{x^2-11}{x^2+7x+6}-\dfrac{x}{x+6}+\dfrac{2}{x+1}}{\dfrac{x+3}{x-3}-\dfrac{5x-2}{x+3}+\dfrac{2x(2x-11)}{x^2-9}} &= \frac{\dfrac{x^2-11-x(x+1)+2(x+6)}{(x+6)(x+1)}}{\dfrac{(x+3)(x+3)-(5x-2)(x-3)+2x(2x-11)}{(x-3)(x+3)}}\\ &= \frac{\dfrac{x^2-11-x^2-x+2x+12}{(x+6)(x+1)}}{\dfrac{x^2+6x+9-5x^2+17x-6+4x^2-22x}{(x-3)(x+3)}}\\ &= \frac{\dfrac{x+1}{(x+6)(x+1)}}{\dfrac{x+3}{(x-3)(x+3)}}\\ &= \frac{1}{x+6}\cdot\frac{x-3}{1}\\ &= \frac{x-3}{x+6}.\end{aligned}$$

Exercises

1. Simplify.

   (a) $\dfrac{4x^3y^2}{6x^4y}$

   (b) $\dfrac{15x^2y}{5x^2-10x}$

   (c) $\dfrac{15+5x}{x^2+7x+12}$

   (d) $\dfrac{2x^2+3x-2}{2x-1}$

   (e) $\dfrac{x^2+2x-15}{9-x^2}$

   (f) $\dfrac{16-y^2}{y^2+2y-24}$

   (g) $\dfrac{2a^3-16}{2a^2+4a+8}$

   (h) $\dfrac{x^2y+y+5x^2+5}{5x+xy}$

   (i) $\dfrac{x^3-12x^2}{x^3-12x^2+2x-24}$

   (j) $\dfrac{a^2+8a+7}{2a^2+a-1}$

   (k) $\dfrac{4-y^2}{y^2-3y-10}$

   (l) $\dfrac{3x^2+x-2}{3x^2+5x+2}$

2. Multiply or/and divide and simplify.

   (a) $\dfrac{14x^3}{15y^2}\cdot\dfrac{25y^3x}{42x^2y}$

   (b) $\dfrac{4a^2b^4}{9x^2y}\cdot\dfrac{27xy^2}{6a^3b}$

   (c) $\dfrac{3x-6}{5x-20}\cdot\dfrac{10x-40}{27x-54}$

   (d) $\dfrac{5x^2-15x}{x^2-8x+15}\cdot\dfrac{25-x^2}{5x^2}$

   (e) $\dfrac{x^3-x^2y}{6x+12y}\cdot\dfrac{3y^2-3y}{3y-3x}$

   (f) $\dfrac{x^2-1}{3x^2+4x+1}\cdot\dfrac{9x^2-1}{3x^2-4x+1}$

   (g) $\dfrac{x^2-3x-10}{x^2-5x}\div\dfrac{x^2-4}{x^2-2x}$

   (h) $\dfrac{t^3-t}{t^2-3t-4}\div\dfrac{t-t^2}{t^2-16}$

(i) $\dfrac{2x^2+8x-42}{3x^2-27} \div \dfrac{2x^2+14x}{3x^2+15x}$

(j) $\dfrac{16y^2-1}{4y^2+3y-1} \div \dfrac{4y^2-7y-2}{y^2-y-2}$

(k) $\dfrac{12t^4+15t^2}{15t^2-t-2} \div \dfrac{4t^3+5t}{9t^2-1}$

(l) $\dfrac{a^4-8a}{a^2-4a-5} \cdot \dfrac{a^2+2a+1}{a^3-a^2-2a} \cdot \dfrac{a^2-5a}{a^2+2a+4}$

(m) $\dfrac{xy+2y^2}{2x^2y+4xy^2} \cdot \dfrac{x^3-xy^2}{x^4-y^4} \cdot \dfrac{x^2y-y^3}{x^2y}$

(n) $(\dfrac{x^2+x}{x^2-25} \cdot \dfrac{x^2-x-20}{3x+12}) \div \dfrac{x^2+3x}{3x^2-27}$

(o) $(\dfrac{x^5y^3}{x^2+13x+30} \cdot \dfrac{x^2+2x-3}{x^3y^2}) \div \dfrac{yx^2-yx}{x^2+10x}$

(p) $\dfrac{6t^2-t-2}{t-1} \cdot \dfrac{3t^2-t-2}{9t^2-4} \div \dfrac{2t+1}{3t+2}$

(q) $(\dfrac{x^2y^5}{x^2-11x+30} \div \dfrac{xy^6}{x^2-7x+10}) \cdot \dfrac{x^2-12x+36}{x^2y-2xy}$

(r) $\dfrac{2x^2-3x-20}{2x^2-7x-30} \div \dfrac{2x^2-5x-12}{4x^2+12x+9} \cdot \dfrac{x^2-36}{4x^2-9}$

3. Add or/and subtract and simplify.

(a) $\dfrac{4x}{x-6} - \dfrac{24}{x-6}$

(b) $3 - \dfrac{2x}{x-1}$

(c) $\dfrac{3y}{y-5} - \dfrac{2y-25}{5-y}$

(d) $\dfrac{3x+1}{x-7} + \dfrac{5x+2}{7-x} - \dfrac{1-2x}{x-7}$

(e) $\dfrac{2x-3}{3x} - \dfrac{4-x}{6}$

(f) $\dfrac{3}{x^2} + \dfrac{2}{5x}$

(g) $\dfrac{x}{x-1} - \dfrac{2}{x^2-1}$

(h) $\dfrac{x+1}{x-1} + \dfrac{x-1}{x+1}$

(i) $\dfrac{2}{y-5} + 2 - \dfrac{3}{y+5}$

(j) $\dfrac{3x+1}{x^2-4} + \dfrac{2}{x-2}$

(k) $\dfrac{5}{4x-12} - \dfrac{3x}{x^2-9}$

(l) $\dfrac{3}{x(x+1)^2} - \dfrac{4}{x^2(x+1)}$

(m) $\dfrac{4x}{x^2-9} + \dfrac{2}{3-x}$

(n) $\dfrac{3x}{x^2-x-2} - \dfrac{2+x}{x^2-1}$

(o) $\dfrac{4t+1}{t-8} - \dfrac{3t+2}{t+4} - \dfrac{49t+4}{t^2-4t-32}$

(p) $\dfrac{x}{x-4} + \dfrac{5}{x+5} - \dfrac{11x-8}{x^2+x-20}$

4. Simplify.

(a) $(\dfrac{1}{x} + \dfrac{1}{y}) \div (x^2-y^2)$

(b) $(\dfrac{x^2}{4} - \dfrac{4}{x^2}) \div (\dfrac{x}{2} - \dfrac{2}{x})$

(c) $\dfrac{2}{1-x^2} \div (\dfrac{1}{1-x} - \dfrac{1}{1+x})$

(d) $\dfrac{9-\dfrac{4}{x^2}}{3+\dfrac{2}{x}}$

(e) $\dfrac{\dfrac{1}{3}-\dfrac{1}{x}}{\dfrac{1}{9}-\dfrac{1}{x^2}}$

(f) $\dfrac{1+\dfrac{x}{y}}{\dfrac{x}{y}-1}$

(g) $\dfrac{x-\dfrac{y^2}{x}}{1+\dfrac{y}{x}}$

(h) $\dfrac{\dfrac{1}{5}-\dfrac{1}{y}}{y-5}$

(i) $\dfrac{\dfrac{1}{x}-\dfrac{1}{1+x}}{\dfrac{1}{1+x}}$

(j) $\dfrac{\dfrac{1}{x}-\dfrac{2}{x-1}}{\dfrac{3}{x}+\dfrac{1}{x-1}}$

(k) $\dfrac{1-\dfrac{7}{y}+\dfrac{12}{y^2}}{1+\dfrac{1}{y}-\dfrac{20}{y^2}}$

8.2 Solving Equations Containing Fractions

To solve an equation containing fractions, we can start by *clearing the denominators* of the fractions; This is accomplished by multiplying each side of the equation by the L.C.D. of all the denominators involved. Once the denominators are gone, we solve for x. *Any solution obtained that causes a denominator of the original equation to vanish (i.e., to be zero) has to be rejected.* These "solutions", which are not really *honest* solutions, are called **extraneous**.

Examples

Solve for x, and decide whether the solution(s) obtained are extraneous or not.

(1)

$$\begin{aligned}
\frac{x}{x+2} - \frac{x}{x-2} &= \frac{x+20}{x^2-4} \quad \Longrightarrow \\
(x-2)(x+2)\left(\frac{x}{x+2} - \frac{x}{x-2}\right) &= (x-2)(x+2)\cdot\frac{x+20}{x^2-4} \quad \Longrightarrow \\
x(x-2) - x(x+2) &= x+20 \quad \Longrightarrow \\
-5x &= 20 \quad \Longrightarrow \\
x &= -4.
\end{aligned}$$

As -4 does not cause any of the denominators to vanish, it is a genion solution.

(2)

$$\begin{aligned}
\frac{1}{x-4} - \frac{3}{x+4} - \frac{6}{5x} &= 0 \quad \Longrightarrow \\
5x(x-4)(x+4)\left(\frac{1}{x-4} - \frac{3}{x+4} - \frac{6}{5x}\right) &= 0 \quad \Longrightarrow \\
5x(x+4) - 15x(x-4) - 6(x-4)(x+4) &= 0 \quad \Longrightarrow \\
-16x^2 + 80x + 96 &= 0 \quad \Longrightarrow \\
x^2 - 5x - 6 &= 0 \quad \Longrightarrow \\
x_1, x_2 &= -1,\ 6.
\end{aligned}$$

No extraneous solution; both are accepted.

(3)

$$\begin{aligned} 1 - \frac{12}{x^2-4} &= \frac{3}{x+2} \quad \Longrightarrow \\ (x-2)(x+2)\left(1 - \frac{12}{x^2-4}\right) &= (x-2)(x+2)\cdot\frac{3}{x+2} \quad \Longrightarrow \\ (x-2)(x+2) - 12 &= 3(x-2) \quad \Longrightarrow \\ x^2 - 3x - 10 &= 0 \quad \Longrightarrow \\ x_1,\ x_2 &= 5, \quad -2. \end{aligned}$$

The only solution is 5, as -2 is extraneous! (why?)

(4)

$$\begin{aligned} \frac{3}{3+x} + \frac{x}{x-3} &= \frac{x^2+9}{x^2-9} \quad \Longrightarrow \\ (x-3)(x+3)\left(\frac{3}{x+3} + \frac{x}{x-3}\right) &= (x-3)(x+3)\cdot\frac{x^2+9}{x^2-9} \quad \Longrightarrow \\ 3(x-3) + x(x+3) &= x^2+9 \quad \Longrightarrow \\ 6x &= 18 \quad \Longrightarrow \\ x &= 3. \end{aligned}$$

No solution at all, as 3 is extraneous!

(5) Double a number minus ten times its reciprocal is 8/3. Find the number.

Solution. Calling the desired number x, we have to solve the equation

$$2x - 10\cdot\frac{1}{x} = \frac{8}{3}.$$

This is easy! The solutions are 3 and $-5/3$.

Exercises

1. Solve the equations.

(a) $\dfrac{3x+1}{8} - \dfrac{1}{4} = \dfrac{x}{2}$

(b) $\dfrac{x-3}{4} - \dfrac{2}{3} = \dfrac{2x-17}{12}$

(c) $\dfrac{3y-1}{4} + \dfrac{2}{3} = \dfrac{y+6}{6}$

(d) $\dfrac{3x+8}{3} + \dfrac{x-1}{5} = \dfrac{1}{15}$

(e) $\dfrac{2x}{x-2} = 1 + \dfrac{4}{x-2}$

(f) $\dfrac{5}{2x} + \dfrac{1}{2} = \dfrac{7x-1}{3x}$

(g) $\dfrac{5}{4x} = \dfrac{1}{x+1} + \dfrac{3}{2x}$

(h) $\dfrac{3y}{y-2} - 3 = \dfrac{2}{5}$

(i) $\dfrac{2t}{t^2-1} + \dfrac{1}{t-1} = \dfrac{2}{t+1}$

(j) $\dfrac{5}{2-x} - \dfrac{3x}{x^2-4} = \dfrac{-7}{x+2}$

(k) $\dfrac{7x}{x^2-x-6} + \dfrac{2}{x-3} = \dfrac{1}{x+2}$

(l) $\dfrac{x}{x+4} = 1 - \dfrac{2}{x}$

(m) $\dfrac{3x}{x-4} = 5 - \dfrac{12}{4-x}$

(n) $\dfrac{3x-1}{3} - \dfrac{2x}{x-1} = x$

(o) $\dfrac{3}{3+t} - \dfrac{t}{3-t} = \dfrac{t^2+9}{t^2-9}$

(p) $\dfrac{x}{x-1} + \dfrac{1}{x} = \dfrac{x^2+1}{x^2-x}$

2. Solve for x.

(a) $\dfrac{1}{x} + x = \dfrac{10}{3}$

(b) $\dfrac{9}{x} + \dfrac{4}{x+4} = 1$

(c) $x + \dfrac{10}{x-7} = 0$

(d) $\dfrac{3x-1}{3} = \dfrac{x}{x-1} - x$

(e) $1 - \dfrac{12}{x^2-4} = \dfrac{3}{x+2}$

(f) $x - \dfrac{6}{2-x} = \dfrac{3x}{x-2}$

(g) $\dfrac{1}{2} - \dfrac{2}{x^2-1} = \dfrac{1}{x+1}$

(h) $x + \dfrac{2}{3} = \dfrac{3x+2}{3x-3}$

(i) $\dfrac{x}{x-4} - \dfrac{7}{x+4} = \dfrac{56}{x^2-16}$

(j) $\dfrac{x}{3-x} - \dfrac{2}{x+3} - \dfrac{6}{x^2-9} = 0$

(k) $\dfrac{2x}{x-3} = \dfrac{10}{x+1} - \dfrac{7x-27}{x-3}$

(l) $\dfrac{1}{x-4} - \dfrac{3}{x+4} = \dfrac{1}{2}$

(m) $\dfrac{x}{x-1}+\dfrac{3}{x}=\dfrac{-3}{x^2-x}$

(n) $2-\dfrac{1}{x}=\dfrac{6}{x+5}$

(o) $\dfrac{3}{x-1}+\dfrac{2}{x+1}=\dfrac{-1}{x-2}$

(p) $\dfrac{x}{x^2+2x+1}+\dfrac{3}{x+1}=\dfrac{1}{x-3}$

(q) $\dfrac{3}{x^2-1}+\dfrac{2x}{x+1}=\dfrac{7x}{x-1}-1$

3. One third of a number is 4 more than its one quarter. Find the number.

4. Find a number whose reciprocal multiplied by twelve is four less than that number.

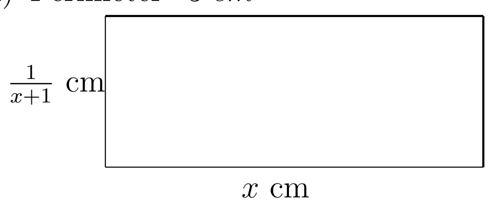
5. Find three consecutive integers such that double the reciprocal of the middle one is the sum of the other two numbers.

6. The sum of a number and its reciprocal is $\dfrac{73}{24}$. Find the number.

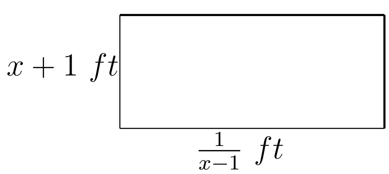
7. The sum of a number and double its reciprocal is $\dfrac{67}{21}$. Find the number.

8. Find x in each of the following geometric objects.

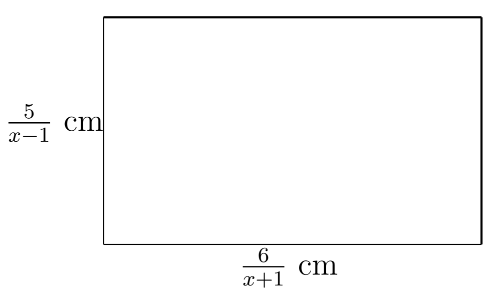
(a) Perimeter=3 *cm*



(b) Perimeter=8 *ft*



(c) Perimeter=$\frac{1}{2}$ *cm*



8.3 Solving Radical Equations

A **radical equation** is an equation in which at least one variable expression is stuck inside a radical, usually a square root. For instance, this is a radical equation:

$$\sqrt{x+1}+5=7,$$

but this is **Not**:

$$2x-3\sqrt{5}=11.$$

A radical equation may contain more than one "radical expression"; here is one example:

$$5\sqrt{3x+1}-2\sqrt{x+8}=4\sqrt{x}.$$

Rematk 1. The "radical(s)" in a radical equation can be of any root, whether square root, cube root, or some other roots, however, most of the examples in what follows deal with square roots.

How to Solve a Radical Equation:

In general, we solve equations by "isolating" the variable; we isolate the variable by "undoing" whatever had been done to it. When you have a variable inside a square root, you undo the root by doing the opposite: *squaring*. For instance, given $\sqrt{x}=3$, you would square both sides:

$$\sqrt{x}=3 \Longrightarrow \left(\sqrt{x}\right)^2=3^2 \Longrightarrow x=9.$$

It should be noted that if there is only one radical expression plus some other terms, prior to squaring, one needs to transform these other terms to the other side of the equation, otherwise the squaring wouldn't remove the radical. We illustrate this in the examples below:

Examples

(1) Solve for x: $\quad \sqrt{x+1}+5=7.$

Solution.

$$\begin{aligned}\sqrt{x+1}+5=7 &\implies \sqrt{x+1}=7-5=2\\ &\implies \left(\sqrt{x+1}\right)^2=2^2\\ &\implies x+1=4\\ &\implies x=3.\end{aligned}$$

(2) Solve for x: $x+\sqrt{x-4}=3.$

Solution.

$$\begin{aligned}x+\sqrt{x-4}=10 &\implies \sqrt{x-4}=10-x\\ &\implies (\sqrt{x-4})^2=(10-x)^2\\ &\implies x-4=100-20x+x^2\\ &\implies x^2-21x+104=0.\end{aligned}$$

The solutions to the last equation are $x=8$ and $x=13$ while by checking these answers into the initial equation, we only accept $x=8$ as the only final solution (see the next example).

Check all Solutions!

One major difficulty with radical equations is that we may have done every step correctly, but our answer may still be wrong. This is because the *very act of squaring the sides can create solutions that never existed before!* To find any extraneous solution (something which in fact is not a solution at all), you should always check your answer(s) by plugging them back into the original equation and making sure that they fit.

(3) Solve for x: $x+\sqrt{x-4}=10.$

Solution.

As we saw in **Exercise 2**, we would find $x=8$ and $x=13$, but now they are needed to be checked by plugging back into the initial equation:

$x=8:\quad 8+\sqrt{8-4}=8+2=10.$

So $x=8$ is accepted whereas:

$x=13:\quad 13+\sqrt{13-4}=13+\sqrt{9}=13+3=16\neq 10,$

shows that $x=13$ has to be rejected.

(4) Solve for x: $\sqrt{2x+10}-x=1.$

Solution.

$$\begin{aligned}\sqrt{2x+10}-x=1 \quad &\Longrightarrow \quad \sqrt{2x+10}=x+1\\ &\Longrightarrow \quad 2x+10=x^2+2x+1\\ &\Longrightarrow \quad x^2-9=0\\ &\Longrightarrow \quad x_1=3,\ \cancel{x_2=-3}.\end{aligned}$$

You can check these two solutions by plugging back into the initial radical equation to see that $x=-3$ does not fit. So the only solution will be $x=3$.

Remark 2. If the equation contains more than one radical expression, we may have to square both sides several times in order to get rid of all the squares. The following examples illustrate this.

(5) Solve $\sqrt{3x-8}+\sqrt{x}=4$.
Solution.

$$\begin{aligned}\sqrt{3x-8}+\sqrt{x}=4 \quad &\Longrightarrow \quad (3x-8)+2\sqrt{3x-8}\sqrt{x}+x=16\\ &\Longrightarrow \quad \sqrt{x(3x-8)}=12-2x\\ &\Longrightarrow \quad x(3x-8)=144-48x+4x^2\\ &\Longrightarrow \quad x^2-40x+144=0\\ &\Longrightarrow \quad x_1=4\ ,\ \cancel{x_2=36}.\end{aligned}$$

(6) Solve $\sqrt[3]{3x+1}=4$
Solution.

$$\begin{aligned}\left(\sqrt[3]{3x+1}\right)^3=4^3 \quad &\Longrightarrow \quad 3x+1=64\\ &\Longrightarrow \quad x=21\end{aligned}$$

Exercises

1. Solve for x.

   (a) $\sqrt{x-3}=4$

   (b) $\sqrt{2x+1}=-3$

   (c) $7-\sqrt{x+1}=5$

   (d) $\sqrt{3x^2-3}+3=9$

   (e) $2\sqrt{2}=\sqrt{x^2+2x}$

   (f) $\sqrt{2x+1}=\sqrt{3x-5}$

   (g) $2\sqrt{x+1}=\sqrt{x^2+4}$

   (h) $3\sqrt{x-1}-2\sqrt{x+4}=0$

   (i) $2\sqrt{x}=\sqrt{3x^2-5x}$

   (j) $2\sqrt{x}=x+1$

   (k) $3\sqrt{x}=2x+1$

   (l) $2\sqrt{2x-1}=x+1$

   (m) $\sqrt{x+1}-x=1$

   (n) $\sqrt{3x+10}+5=2x$

   (o) $\sqrt{x^2-4x}-3=x$

   (p) $\sqrt{5x+1}-x=1$

   (q) $x-\sqrt{x^2-5}=1$

   (r) $\sqrt{3x+7}-2(x-1)=0$

   (s) $\sqrt{\dfrac{x^2+2x}{x-2}}=\sqrt{3}$

2. Solve the equations for the given unknowns.

   (a) $\sqrt{x-5}+\sqrt{x}=5$

   (b) $\sqrt{t-8}=2-\sqrt{t}$

   (c) $\sqrt{x+1}+1=\sqrt{2x}$

   (d) $\sqrt{s}-\sqrt{s-4}=1$

   (e) $\sqrt{y}+\sqrt{y+7}=1$

   (f) $\sqrt{2x+4}+2=2\sqrt{x}$

   (g) $\sqrt{8t+33}-3=2\sqrt{2t}$

   (h) $\sqrt{2x+1}-\sqrt{2x-4}=1$

   (i) $\sqrt{3n+1}=1+\sqrt{3n-2}$

   (j) $\sqrt{2x-4}=\sqrt{3x+4}-2$

   (k) $\sqrt{x-4}-\sqrt{x+1}=1$

   (l) $\sqrt{4x+2}+\sqrt{2x}=\sqrt{2}$

   (m) $\sqrt{2x+1}-\sqrt{x-10}=2\sqrt{3}$

3. Solve the equations for the given unknowns.

(a) $(2\sqrt{x}-3)(2\sqrt{x}+3)=7$

(b) $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}=\dfrac{\sqrt{x}+2}{\sqrt{x}+7}$

(c) $3-\sqrt{x}=\dfrac{2\sqrt{x}+6}{\sqrt{x}+3}$

(d) $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$

(e) $\dfrac{1}{\sqrt{x-1}}+\sqrt{x-1}=2$

(f) $\sqrt{x-1}+\sqrt{x}=\dfrac{2}{\sqrt{x}}$

(g) $\sqrt{4x+5}-\sqrt{x}=\sqrt{x+3}$

(h) $\sqrt{x+3}-\sqrt{x+8}+\sqrt{x}=0$

8.4 Final Answer to Exercises

Section 8.1

1. (a) $\frac{2y}{3x}$
   (b) $\frac{3xy}{x-2}$
   (c) $\frac{5}{x+4}$
   (d) $x+2$
   (e) $\frac{-(x+5)}{x+3}$
   (f) $\frac{-(4+y)}{y+6}$
   (g) $a-2$
   (h) $\frac{x^2+1}{x}$
   (i) $\frac{x^2}{x^2+2}$
   (j) $\frac{a+7}{2a-1}$
   (k) $\frac{2-y}{y-5}$
   (l) $\frac{3x-2}{3x+2}$

2. (a) $\frac{5x^2}{9}$
   (b) $\frac{2b^3y}{ax}$
   (c) $\frac{2}{9}$
   (d) $\frac{-(5+x)}{x}$
   (e) $\frac{-x^2y(y-1)}{6(x+2y)}$
   (f) 1
   (g) 1
   (h) $-(t+4)$
   (i) $\frac{x+5}{x+3}$
   (j) 1
   (k) $\frac{3t(3t-1)}{5t-2}$
   (l) a
   (m) $\frac{(x+y)(x-y)}{2x^2(x^2+y^2)}$
   (n) $\frac{(x+1)(x-3)}{x+5}$
   (o) x^2
   (p) $3t+2$
   (q) $\frac{x-6}{y^2}$
   (r) $\frac{(x+6)}{(2x-3)}$

3. (a) 4
   (b) $\frac{x-3}{x-1}$
   (c) 5
   (d) $\frac{-2}{x-7}$
   (e) $\frac{x^2-6}{6x}$
   (f) $\frac{15+2x}{5x^2}$

(g) $\dfrac{x+2}{x+1}$

(h) $\dfrac{2x^2+2}{x^2-1}$

(i) $\dfrac{2y^2-y-25}{y^2-25}$

(j) $\dfrac{5x+5}{x^2-4}$

(k) $\dfrac{15-7x}{4(x^2-9)}$

(l) $\dfrac{-x-4}{x^2(x+1)^2}$

(m) $\dfrac{2}{x+3}$

(n) $\dfrac{2x^2-3x+4}{(x-1)(x+1)(x-2)}$

(o) $\dfrac{t-2}{t+4}$

(p) $\dfrac{x+3}{x+5}$

4. (a) $\dfrac{1}{xy(x-y)}$

(b) $\dfrac{x^2+4}{2x}$

(c) $\dfrac{1}{x}$

(d) $\dfrac{3x-2}{x}$

(e) $\dfrac{3x}{x+3}$

(f) $\dfrac{y+x}{x-y}$

(g) $x-y$

(h) $\dfrac{1}{5y}$

(i) $\dfrac{1}{x}$

(j) $\dfrac{-x-1}{4x-3}$

(k) $\dfrac{y-3}{y+5}$

Section 8.2

1. (a) $x=-1$

(b) $x=0$

(c) $y=1$

(d) $x=-2$

(e) No Solution.

(f) $x=\dfrac{17}{11}$

(g) $x=-\dfrac{1}{5}$

(h) $y=17$

(i) $t=-3$

(j) $x=-24$

(k) $x=-\dfrac{7}{8}$

(l) $x=4$

(m) No Solution.

(n) $x=\dfrac{1}{7}$

(o) No Solution.

(p) $x=2$

2.

(a) $x = \dfrac{1}{3}$ or $x = 3$

(b) $x = -3$ or $x = 12$

(c) $x = 2$ *or* $x = 5$

(d) $x = \dfrac{5 \pm \sqrt{19}}{6}$

(e) $x = 5$

(f) $x = 3$

(g) $x = 3$

(h) $x = -\dfrac{2}{3}$ or $x = 2$

(i) $x = 7$

(j) $x = -5$ or $x = 0$

(k) $x = -\dfrac{1}{9}$ or $x = \dfrac{25}{9}$

(l) $x = -2 \pm \sqrt{52}$

(m) $x = -3$

(n) $x = -\dfrac{5}{2}$ or $x = 1$

(o) $x = \dfrac{9 \pm \sqrt{105}}{12}$

(p) $x = \dfrac{11 \pm \sqrt{241}}{6}$

(q) $x = \dfrac{-9 \pm \sqrt{113}}{8}$

3. 48

4. -2 *or* 6

5. $-2, \; -1, \; 0$ or $0, \; 1, \; 2$

6. 3/8 or 8/3

7. 6/7 or 7/3

8. (a) $x = 1$

   (b) $x = 2$

   (c) $x = 22 + \sqrt{481}$

Section 8.3

1. (a) $x = 19$

   (b) No Solution.

   (c) $x = 3$

   (d) $x = \pm\sqrt{13}$

   (e) $x = -4$ or $x = 2$

   (f) $x = 6$

   (g) $x = 0$ or $x = 4$

   (h) $x = 5$

   (i) $x = 0$ or $x = 3$

   (j) $x = 1$

   (k) $x = 1$ or $x = \dfrac{1}{4}$

   (l) $x = 1$ or $x = 5$

   (m) $x = 0$ or $x = -1$

(n) $x = 5$
(o) $x = \dfrac{-9}{10}$
(p) $x = 0$ or $x = 3$
(q) $x = 3$
(r) $x = 3$
(s) No Solution.

2. (a) $x = 9$
(b) No Solution.
(c) $x = 8$
(d) $s = \dfrac{25}{4}$
(e) No Solution.
(f) $x = 16$
(g) $t = 2$
(h) $x = 4$
(i) $n = 1$
(j) $x = 4$ or $x = 20$
(k) No Solution.
(l) No Solution.
(m) $x = 13$ or $x = 37$

3. (a) $x = 4$
(b) $x = 9$
(c) $x = 1$
(d) $x = 25$
(e) $x = 2$
(f) $x = \dfrac{4}{3}$
(g) $x = 1$
(h) $x = 1$